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In any Δ ABC, the following relationship holds :

$$\sum_{cyc} \sqrt[2023]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}} + \left(\frac{R}{2r}\right)^3 \geq 1 + \sum_{cyc} \sqrt[2023]{\frac{a}{b+c}}$$

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$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Now, } (b+c)^2 \overset{?}{\geq} 32Rr\cos^2\frac{A}{2} \overset{\text{via (i)}}{=} 8r(r_b+r_c) = 8r^2 s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$= 8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \overset{?}{\geq} 0$$

$$\Leftrightarrow (b+c-2a)^2 \overset{?}{\geq} 0 \to \text{true} \therefore b+c \geq \sqrt{32Rr}.\cos\frac{A}{2} \text{ and analogs} \Rightarrow$$

$$\sum_{cyc} \sqrt[2023]{\frac{a}{b+c}} \leq \sum_{cyc} \sqrt[2023]{\frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{\sqrt{32Rr}.\cos\frac{A}{2}}} = \sqrt[4046]{\frac{R}{2r}}.\sum_{cyc} \sqrt[2023]{\sin\frac{A}{2}}$$

$$\overset{\text{Jensen}}{\leq} \sqrt[4046]{\frac{R}{2r}}.3.\sqrt[2023]{\frac{1}{2}}$$

$$\left(\because f''(x) = -\frac{2023\sin^2\frac{A}{2} + 2022\cos^2\frac{A}{2}}{16370116\left(\sin\frac{A}{2}\right)^{\frac{4045}{2023}}} < 0 \text{ where } f(x) = \sqrt[2023]{\sin\frac{x}{2}}\ \forall x \in (0,\pi)\right)$$

$$\therefore \boxed{\sum_{cyc} \sqrt[2023]{\frac{a}{b+c}} \leq 3.\sqrt[2023]{\frac{1}{2}}.\sqrt[4046]{\frac{R}{2r}}} \to (1)$$

$$\text{Also, } \frac{\frac{r}{R}}{\frac{9}{2}-\frac{r}{R}} \overset{?}{\geq} \frac{r^2}{2R^2} \Leftrightarrow \frac{2R}{9R-2r} \overset{?}{\geq} \frac{r}{2R} \Leftrightarrow 4R^2 - 9Rr + 2r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(4R-r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \therefore \frac{\frac{r}{R}}{\frac{9}{2}-\frac{r}{R}} \geq \frac{r^2}{2R^2} \to \text{(ii)}$$

$$\text{Again, } \sum_{cyc}\sqrt[2023]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}} \overset{A-G}{\geq} 3.\sqrt[3]{\sqrt[2023]{\frac{\prod_{cyc}\sin\frac{A}{2}}{\prod_{cyc}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}}}$$

$$= 3.\sqrt[6069]{\frac{\frac{r}{4R}}{\left(\sum_{cyc}\sin\frac{A}{2}\right)\left(\sum_{cyc}\sin\frac{A}{2}\sin\frac{B}{2}\right)-\frac{r}{4R}}} \geq 3.\sqrt[6069]{\frac{\frac{r}{4R}}{\frac{\left(\sum_{cyc}\sin\frac{A}{2}\right)^3}{3}-\frac{r}{4R}}}$$

$$\overset{Jensen}{\geq} 3.\sqrt[6069]{\frac{\frac{r}{4R}}{\frac{\left(\frac{3}{2}\right)^3}{3}-\frac{r}{4R}}} \overset{via\ (ii)}{\geq} 3.\sqrt[6069]{\frac{r^2}{2R^2}} = 3.\sqrt[6069]{\frac{4r^2}{R^2}\cdot\frac{1}{2^3}}$$

$$\therefore \boxed{\sum_{cyc}\sqrt[2023]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}} \geq 3.\sqrt[2023]{\frac{1}{2}}.\sqrt[6069]{\frac{4r^2}{R^2}}} \rightarrow (2) \therefore (1),(2) \Rightarrow \text{in order to prove :}$$

$$\sum_{cyc}\sqrt[2023]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}}+\frac{R}{2r} \geq 1+\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}, \text{it suffices to prove :}$$

$$3.\sqrt[2023]{\frac{1}{2}}.\sqrt[6069]{\frac{4r^2}{R^2}}+\frac{R}{2r}-1 \geq 3.\sqrt[2023]{\frac{1}{2}}.\sqrt[4046]{\frac{R}{2r}}$$

$$\Leftrightarrow \frac{R}{2r}-1 \geq 3.\sqrt[2023]{\frac{1}{2}}.\left(\left(\frac{R}{2r}\right)^{\frac{1}{4046}}-\left(\frac{2r}{R}\right)^{\frac{2}{6069}}\right)$$

$$\Leftrightarrow \left(\left(\frac{R}{2r}\right)^{\frac{1}{12138}}\right)^{12138}-1 \geq 3.\sqrt[2023]{\frac{1}{2}}.\left(\left(\left(\frac{R}{2r}\right)^{\frac{1}{12138}}\right)^3-\left(\left(\frac{2r}{R}\right)^{\frac{1}{12138}}\right)^4\right)$$

$$\Leftrightarrow \boxed{t^{12138}-1 \overset{(*)}{\geq} 3.\sqrt[2023]{\frac{1}{2}}.\left(t^3-\frac{1}{t^4}\right)} \left(t=\left(\frac{R}{2r}\right)^{\frac{1}{12138}} \geq 1\right)$$

$$\because 2^{\frac{1}{2023}} > 1 \text{ and } \because t^3-\frac{1}{t^4} \geq 0 \therefore 3.\sqrt[2023]{\frac{1}{2}}.\left(t^3-\frac{1}{t^4}\right) \leq 3\left(t^3-\frac{1}{t^4}\right) \overset{?}{\leq} t^{12138}-1$$

$$\Leftrightarrow \boxed{t^{12138}-3t^3+\frac{3}{t^4}-1 \overset{?}{\geq} 0 \overset{(**)}{\geq} 0}$$

$$\text{Let } f(t)=t^{12138}-3t^3+\frac{3}{t^4}-1\ \forall\, t \geq 1 \text{ and then : } f'(t)=\frac{12138t^{12142}-9t^7-12}{t^5}$$

$$=\frac{9\left(t^{12142}-t^7\right)+12\left(t^{12142}-1\right)+12117t^{12142}}{t^5} \geq \frac{12117t^{12142}}{t^5}$$

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$$\left(\because t=\left(\frac{R}{2r}\right)^{\frac{1}{12138}}\overset{\text{Euler}}{\geq}1\right)>0\Rightarrow f(t)\text{ is }\uparrow\forall t\geq 1\Rightarrow f(t)\geq f(1)=0\Rightarrow(**)\Rightarrow(*)$$

$$\text{is true}\therefore\frac{R}{2r}\geq 1+\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}-\sum_{cyc}\sqrt[2023]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}}\ \textit{and}\because\left(\frac{R}{2r}\right)^3\geq\frac{R}{2r}\therefore$$

$$\left(\frac{R}{2r}\right)^3\geq 1+\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}-\sum_{cyc}\sqrt[2023]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}}\therefore\sum_{cyc}\sqrt[2023]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}}+\left(\frac{R}{2r}\right)^3$$

$$\geq 1+\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}\ \forall\Delta ABC,''='' \text{ iff }\Delta ABC\text{ is equilateral (QED)}$$