ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\sqrt[2023]{\frac{h_a}{h_b+h_c}}+\left(\frac{R}{2r}\right)^3\geq 1+\sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}$$

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$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Now, }(b+c)^2\overset{?}{\geq}32Rr\cos^2\frac{A}{2}\overset{\text{via (i)}}{=}8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$=8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)}=4a(b+c-a)\Leftrightarrow(b+c)^2+4a^2-4a(b+c)\overset{?}{\geq}0$$

$$\Leftrightarrow(b+c-2a)^2\overset{?}{\geq}0\to\text{true}\therefore b+c\geq\sqrt{32Rr}.\cos\frac{A}{2}\ \textit{and analogs}\Rightarrow$$

$$\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}\leq\sum_{cyc}\sqrt[2023]{\frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{\sqrt{32Rr}.\cos\frac{A}{2}}}=\sqrt[4046]{\frac{R}{2r}}.\sum_{cyc}\sqrt[2023]{\sin\frac{A}{2}}$$

$$\overset{\text{Jensen}}{\leq}\sqrt[4046]{\frac{R}{2r}}.3.\sqrt[2023]{\frac{1}{2}}$$

$$\left(\because f''(x)=-\frac{2023\sin^2\frac{A}{2}+2022\cos^2\frac{A}{2}}{16370116\left(\sin\frac{A}{2}\right)^{\frac{4045}{2023}}}<0\text{ where }f(x)=\sqrt[2023]{\sin\frac{x}{2}}\ \forall\,x\in(0,\pi)\right)$$

$$\therefore\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}\leq 3.\sqrt[2023]{\frac{1}{2}}.\sqrt[4046]{\frac{R}{2r}}\to(1)$$

Implementing (1) on a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$ whose *area as a* consequence of trivial *calculations* $=\frac{F}{3}$, *we arrive at* : $\sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}\leq$

$$3.\sqrt[2023]{\frac{1}{2}}.\sqrt[4046]{\frac{\left(\frac{\frac{2m_a}{3}.\frac{2m_b}{3}.\frac{2m_c}{3}}{\frac{4F}{3}}\right)}{\left(\frac{\frac{2F}{3}}{\frac{m_a}{3}+\frac{m_b}{3}+\frac{m_c}{3}}\right)}}\left(\because\frac{R}{2r}=\frac{\left(\frac{abc}{4F}\right)}{\left(\frac{2F}{s}\right)}\right)=3.\sqrt[2023]{\frac{1}{2}}.\sqrt[4046]{\frac{m_am_bm_c\left(\sum_{cyc}m_a\right)}{9F^2}}$$

$$\overset{m_a m_b m_c \le \frac{Rs^2}{2}\text{ and Leuenberger + Euler}}{\le} 3.\sqrt[2023]{\frac{1}{2}}.\sqrt[4046]{\frac{\frac{Rs^2}{2}\left(\frac{9R}{2}\right)}{9r^2s^2}}$$

$$\therefore \sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}} \le 3.\sqrt[2023]{\frac{1}{2}}.\sqrt[2023]{\frac{R}{2r}} \to (2)$$

$$\text{Again}, \sum_{cyc}\sqrt[2023]{\frac{h_a}{h_b+h_c}} \overset{A-G}{\ge} 3.\sqrt[6069]{\frac{bc.ca.ab}{abc(a+b)(b+c)(c+a)}}$$

$$= 3.\sqrt[6069]{\frac{4Rrs}{2s(s^2+2Rr+r^2)}} \overset{\text{Gerretsen}}{\ge} 3.\sqrt[6069]{\frac{2Rr}{4R^2+6Rr+4r^2}} \overset{\text{Euler}}{\ge} 3.\sqrt[6069]{\frac{2Rr}{4R^2+3R^2+R^2}}$$

$$= 3.\sqrt[6069]{\frac{8r}{4R}.\frac{1}{8}} \therefore \sum_{cyc}\sqrt[2023]{\frac{h_a}{h_b+h_c}} \ge 3.\sqrt[2023]{\frac{1}{2}}.\sqrt[6069]{\frac{2r}{R}} \to (3) \therefore (2),(3) \Rightarrow \text{in order}$$

$$\text{to prove}: \sum_{cyc}\sqrt[2023]{\frac{h_a}{h_b+h_c}} + \left(\frac{R}{2r}\right)^3 - 1 \ge \sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}, \text{ it suffices to prove}:$$

$$3.\sqrt[2023]{\frac{1}{2}}.\sqrt[6069]{\frac{2r}{R}} + \left(\frac{R}{2r}\right)^3 - 1 \ge 3.\sqrt[2023]{\frac{1}{2}}.\sqrt[2023]{\frac{R}{2r}}$$

$$\Leftrightarrow \left(\frac{R}{2r}\right)^3 - 1 \ge 3.\sqrt[2023]{\frac{1}{2}}.\left(\sqrt[2023]{\frac{R}{2r}} - \sqrt[6069]{\frac{2r}{R}}\right) \text{ and to prove it, it sufices to prove}:$$

$$\left(\frac{R}{2r}\right)^3 - 1 \ge 3\left(\sqrt[2023]{\frac{R}{2r}} - \sqrt[6069]{\frac{2r}{R}}\right)\left(\because \sqrt[2023]{\frac{1}{2}} < 1\right)$$

$$\Leftrightarrow t^{18207} - 1 \ge 3\left(t^3 - \frac{1}{t}\right)\left(\text{where } t = \sqrt[6069]{\frac{R}{2r}} \ge 1\right) \Leftrightarrow t^{18208} - t \overset{(*)}{\ge} 3(t^4 - 1)$$

$$\text{Let } f(t) = t^{18208} - t - 3t^4 + 3\ \forall\, t \ge 1 \text{ and then}: f'(t) = 18208t^{18207} - 12t^3 - 1$$

$$= 12t^3(t^{18204} - 1) + (t^{18207} - 1) \overset{t\ge 1}{\ge} 0 \Rightarrow f(t) \text{ is} \uparrow \forall\, t \ge 1 \Rightarrow f(t) \ge f(1) = 0$$

$$\Rightarrow (*) \text{ is true} \therefore \sum_{cyc}\sqrt[2023]{\frac{h_a}{h_b+h_c}} + \left(\frac{R}{2r}\right)^3 \ge 1 + \sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}$$

$\forall\, \Delta\, ABC$, " = " iff $\Delta\, ABC$ is equilateral (QED)

Proof of $m_a m_b m_c \le \frac{Rs^2}{2}$

$$m_a^2 m_b^2 m_c^2 = \frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$\overset{(1)}{=}\frac{1}{64}\left\{-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right\}$$

$$\text{Now, }\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3-3\left(2a^2b^2c^2+\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore \sum_{cyc}a^6\overset{(2)}{=}\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4=\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\overset{(3)}{=}$$

$$\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-3a^2b^2c^2\ \therefore (1),(2),(3)\Rightarrow m_a^2m_b^2m_c^2$$

$$=\frac{1}{64}\begin{pmatrix}-4\left(\sum_{cyc}a^2\right)^3-12a^2b^2c^2+12\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\\ +6\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-18a^2b^2c^2+3a^2b^2c^2\end{pmatrix}$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\left(\sum_{cyc}ab\right)^2-16Rrs^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left\{\begin{matrix}-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2\\ -576Rrs^2(s^2-4Rr-r^2)-432R^2r^2s^2\end{matrix}\right\}$$

$$=\frac{1}{16}\{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\}$$

$$\leq\frac{R^2s^4}{4}\Leftrightarrow$$

$$s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3 \overset{(\bullet)}{\leq} 0$$

$$\text{Now, LHS of } (\bullet) \overset{\text{Gerretsen}}{\leq} -s^4(8Rr - 36r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4)$$
$$-r^3(4R + r)^3 \overset{?}{\leq} 0$$
$$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \underbrace{\overset{?}{\geq}}_{(\bullet\bullet)} 20rs^4$$

$$\text{Now, LHS of } (\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(a)} s^2(16Rr - 5r^2)(8R - 16r)$$
$$+s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \textit{ and}$$
$$\text{RHS of } (\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\leq}}_{(b)} 20rs^2(4R^2 + 4Rr + 3r^2)$$

$(a), (b) \Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :

$$s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3$$
$$\geq 20rs^2(4R^2 + 4Rr + 3r^2)$$
$$\Leftrightarrow s^2(108R^2 - 256Rr + 53r^2) + r(4R + r)^3 \geq 0$$
$$\Leftrightarrow s^2(108R^2 - 256Rr + 80r^2) + r(4R + r)^3 \overset{(\bullet\bullet\bullet)}{\geq} 27r^2s^2$$

$$\text{Now, LHS of } (\bullet\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(c)} (108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R + r)^3$$
$$\textit{and } \text{RHS of } (\bullet\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\leq}}_{(d)} 27r^2(4R^2 + 4Rr + 3r^2)$$

$(c), (d) \Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$$(108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R + r)^3 \geq 27r^2(4R^2 + 4Rr + 3r^2)$$
$$\Leftrightarrow 224t^3 - 587t^2 + 308t - 60 \geq 0 \left(\text{where } t = \frac{R}{r}\right)$$
$$\Leftrightarrow (t - 2)\big((t - 2)(224t + 309) + 648\big) \geq 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet)$$
$$\Rightarrow (\bullet) \text{ is true} \Rightarrow m_a^2m_b^2m_c^2 \leq \frac{R^2s^4}{4} \Rightarrow m_am_bm_c \leq \frac{Rs^2}{2} \text{ (QED)}$$