ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationships holds :

$$\frac{m_a}{r_a}+\frac{m_b}{r_b}+\frac{m_c}{r_c}+\frac{R^2}{4r^2}\geq 1+\frac{r_a}{m_a}+\frac{r_b}{m_b}+\frac{r_c}{m_c}$$

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We have $m_a\geq\frac{b+c}{2}\cos\frac{A}{2}=\frac{2(s-a)+a}{2}.\sqrt{\frac{s(s-a)}{bc}}\geq$

$$\geq\sqrt{2(s-a)a}.\sqrt{\frac{a(s-a)}{4Rr}}=\frac{a(s-a)}{\sqrt{2Rr}}\Rightarrow\frac{m_a}{r_a}\geq\frac{a(s-a)^2}{F\sqrt{2Rr}}\ \text{(and analogs)}$$

$$\Rightarrow\frac{m_a}{r_a}+\frac{m_b}{r_b}+\frac{m_c}{r_c}\geq\frac{a(s-a)^2+b(s-b)^2+c(s-c)^2}{F\sqrt{2Rr}}$$

$$=\frac{s^2.2s-2s.2(s^2-r^2-4Rr)+2s(s^2-3r^2-6Rr)}{F\sqrt{2Rr}}=\frac{4R-2r}{\sqrt{2Rr}}.$$

Now, since $m_a\geq h_a$ (and analogs), then we have

$$\frac{r_a}{m_a}+\frac{r_b}{m_b}+\frac{r_c}{m_c}\leq\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}=\sum_{cyc}\frac{a}{2(s-a)}=\frac{1}{2}\sum_{cyc}\left(\frac{s}{s-a}-1\right)=$$

$$=\frac{1}{2}\left(\frac{4R+r}{r}-3\right)=\frac{2R}{r}-1.$$

From these results, it suffices to prove that

$$\frac{4R-2r}{\sqrt{2Rr}}+\frac{R^2}{4r^2}\geq\frac{2R}{r}\overset{x=\sqrt{\frac{R}{2r}}}{\Leftrightarrow}4x-\frac{1}{x}+x^4\geq 4x^2\Leftrightarrow x^5-4x^3+4x^2-1\geq 0$$

$$\Leftrightarrow (x-1)[x(x-1)(x^2+2x-1)+1]\geq 0,$$

which is true by Euler's inequality

$x=\sqrt{\frac{R}{2r}}\geq 1$. Equality holds iff ΔABC is equilateral.