In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\sqrt{\frac{w_a}{g_a+g_b}}+\frac{R^3}{8r^3}\ge 1+\sum_{cyc}\sqrt{\frac{g_a}{w_b+w_c}}$$

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$$\text{Triangle inequality} \Rightarrow g_a \le AI + r \overset{?}{\le} w_a \Leftrightarrow \frac{r}{\sin\frac{A}{2}}+r\overset{?}{\le}\frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow \frac{r}{\sin\frac{A}{2}}+r\overset{?}{\le}\frac{8Rrs\cos\frac{A}{2}}{4R(b+c)\sin\frac{A}{2}\cos\frac{A}{2}} \Leftrightarrow \frac{1}{\sin\frac{A}{2}}+1\overset{?}{\le}\frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow \frac{1}{\sin\frac{A}{2}}+1\overset{?}{\le}\frac{a}{(b+c)\sin\frac{A}{2}}+\frac{1}{\sin\frac{A}{2}} \Leftrightarrow (b+c)\sin\frac{A}{2}\overset{?}{\le} a$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2}\overset{?}{\le}4R\sin\frac{A}{2}\cos\frac{A}{2}\Leftrightarrow \cos\frac{B-C}{2}\overset{?}{\le}1\to \text{true}$$

$$\therefore g_a \le w_a \text{ and analogs} \to (1)$$

$$\text{Now, }\sum_{cyc}\frac{a}{b}\overset{CBS}{\le}\sqrt{\sum_{cyc}a^2}\cdot\sqrt{\sum_{cyc}\frac{b^2c^2}{16R^2r^2s^2}}\overset{\text{Leibnitz + Goldstone}}{\le}\sqrt{9R^2}\cdot\sqrt{\frac{4R^2s^2}{16R^2r^2s^2}}\Rightarrow\sum_{cyc}\frac{a}{b}\le\frac{3R}{2r}\to(2)$$

$$\text{Also, }\sum_{cyc}\frac{1}{\sin\frac{A}{2}}=\sum_{cyc}\sqrt{\frac{bc(s-a)}{r^2s}}\overset{CBS}{\le}\frac{1}{r.\sqrt{s}}\cdot\sqrt{\sum_{cyc}(s-a)}\cdot\sqrt{s^2+4Rr+r^2}\overset{Geretsen}{\le}$$

$$\frac{1}{r}\cdot\sqrt{4R^2+4Rr+3r^2+4Rr+r^2}\Rightarrow\sum_{cyc}\frac{1}{\sin\frac{A}{2}}\le\frac{2(R+r)}{r}\to(3)$$

$$\text{Now, via (1), }\sum_{cyc}\sqrt{\frac{w_a}{g_a+g_b}}\ge\sum_{cyc}\sqrt{\frac{w_a}{w_a+w_b}}\ge\sum_{cyc}\sqrt{\frac{h_a}{m_a+m_b}}\overset{Panaitopol}{\ge}$$

$$\sum_{cyc}\sqrt{\frac{h_a}{\frac{R}{2r}(h_a+h_b)}}=\sqrt{\frac{2r}{R}}\cdot\sum_{cyc}\sqrt{\frac{\frac{bc}{2R}}{\frac{bc+ca}{2R}}}=\sqrt{\frac{2r}{R}}\cdot\sum_{cyc}\frac{1}{\sqrt{1+\frac{a}{b}}}\overset{\text{Bergstrom + CBS and via (2)}}{\ge}\sqrt{\frac{2r}{R}}\cdot\frac{9}{\sqrt{3}.\sqrt{3+\frac{3R}{2r}}}$$

$$\therefore \boxed{\sum_{cyc}\sqrt{\frac{w_a}{g_a+g_b}}\ge\frac{3}{\sqrt{t(t+1)}}}\to(\text{i})\left(\text{where } t=\frac{R}{2r}\overset{Euler}{\ge}1\right)$$

$$\text{Again, via (1)}, \sum_{cyc}\sqrt{\frac{g_a}{w_b+w_c}} \leq \sum_{cyc}\sqrt{\frac{w_a}{w_b+w_c}} \leq \sum_{cyc}\sqrt{\frac{w_a}{h_b+h_c}}$$

$$= \sum_{cyc}\sqrt{\frac{2bc.\cos\frac{A}{2}}{(b+c)\frac{(b+c).4R\cos\frac{A}{2}\sin\frac{A}{2}}{2R}}} = \frac{1}{2}\sum_{cyc}\sqrt{\frac{4bc}{(b+c)^2}\cdot\frac{1}{\sin\frac{A}{2}}} \overset{\substack{A-G\\ and\\ CBS}}{\leq} \frac{\sqrt{3}}{2}\cdot\sqrt{\sum_{cyc}\frac{1}{\sin\frac{A}{2}}}$$

$$\overset{via\ (3)}{\leq} \frac{\sqrt{3}}{2}\cdot\sqrt{\frac{2(R+r)}{r}} \therefore \boxed{\sum_{cyc}\sqrt{\frac{g_a}{w_b+w_c}} \leq \sqrt{3}.\sqrt{t+\frac{1}{2}}} \rightarrow (ii)$$

$$\therefore (i),(ii) \Rightarrow \text{in order to prove}: \sum_{cyc}\sqrt{\frac{w_a}{g_a+g_b}} + \frac{R^3}{8r^3} \geq 1 + \sum_{cyc}\sqrt{\frac{g_a}{w_b+w_c}},$$

$$\text{it suffices to prove}: \boxed{\frac{3}{\sqrt{t(t+1)}} + t^3 - 1 \overset{(*)}{\geq} \sqrt{3}.\sqrt{t+\frac{1}{2}}}$$

$$\text{Let } f(t) = \frac{3}{\sqrt{t(t+1)}} + t^3 - 1 - \sqrt{3}.\sqrt{t+\frac{1}{2}}\ \forall\, t \geq 1$$

$$\therefore f'(t) = 3t^2 - \frac{\sqrt{3}}{2.\sqrt{t+\frac{1}{2}}} - \frac{3(2t+1)}{2.\big(t(t+1)\big)^{\frac{3}{2}}} \text{ and}$$

$$f''(t) = 6t + \frac{\sqrt{3}}{4(t+1)^{\frac{3}{2}}} + \frac{9(2t+1)^2}{4t(t+1)\big(t(t+1)\big)^{\frac{3}{2}}} - \frac{3}{\big(t(t+1)\big)^{\frac{3}{2}}}$$

$$= 6t + \frac{\sqrt{3}}{4(t+1)^{\frac{3}{2}}} + \frac{3}{\big(t(t+1)\big)^{\frac{3}{2}}}\cdot\frac{3(2t+1)^2 - 4t(t+1)}{4t(t+1)}$$

$$= 6t + \frac{\sqrt{3}}{4(t+1)^{\frac{3}{2}}} + \frac{3}{\big(t(t+1)\big)^{\frac{3}{2}}}\cdot\frac{8t^2+8t+3}{4t(t+1)} > 0 \Rightarrow f''(t) > 0 \Rightarrow f'(t) \text{ is } \uparrow$$

$$\text{on } [1,\infty) \Rightarrow f'(t) \geq f'(0) \approx 0.701903 > 0 \Rightarrow f(t) \text{ is } \uparrow \text{ on } [1,\infty)$$

$$\Rightarrow f(t) \geq f(1) = 0 \Rightarrow \frac{3}{\sqrt{t(t+1)}} + t^3 - 1 - \sqrt{3}.\sqrt{t+\frac{1}{2}} \geq 0\ \forall\, t \geq 1 \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\sqrt{\frac{w_a}{g_a+g_b}} + \frac{R^3}{8r^3} \geq 1 + \sum_{cyc}\sqrt{\frac{g_a}{w_b+w_c}}\ \forall\, \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$