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In $any\ \Delta\ ABC$ and for all $n \ge 2$, the following relationship holds :

$$\sum_{cyc} \sqrt[2011]{\frac{a(b+c)}{bc}} + \frac{R^n}{r^n} \ge 2^n + \sum_{cyc} \sqrt[2011]{\frac{m_a(m_b+m_c)}{m_b m_c}}$$

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$$\sum_{cyc} \sqrt[2011]{\frac{a(b+c)}{bc}} \ge \sum_{cyc} \sqrt[2011]{\frac{2\sqrt{(s-b)(s-c)}.(b+c)}{bc}} \overset{A-G}{\ge}$$

$$\sum_{cyc} \sqrt[2011]{\frac{4.\sqrt{(s-b)(s-c)}}{\sqrt{bc}}} = \sum_{cyc} \sqrt[2011]{4\sin\frac{A}{2}} \overset{A-G}{\ge} 3\sqrt[6033]{64\prod_{cyc}\sin\frac{A}{2}} = 3.\sqrt[6033]{64.\frac{r}{4R}}$$

$$\therefore \boxed{\sum_{cyc} \sqrt[2011]{\frac{a(b+c)}{bc}} \ge 3.\sqrt[6033]{\frac{16r}{R}}} \to (1)$$

$$\text{Again, } \sum_{cyc} \sqrt[2011]{\frac{m_a(m_b+m_c)}{m_b m_c}} = \sum_{cyc} \sqrt[2011]{\frac{m_a}{m_c} + \frac{m_a}{m_b}} \overset{Panaitopol}{\le} \sum_{cyc} \sqrt[2011]{\frac{\frac{Rs}{a}}{\frac{2rs}{c}} + \frac{\frac{Rs}{a}}{\frac{2rs}{b}}}$$

$$= \sum_{cyc} \sqrt[2011]{\frac{R}{2r}.\frac{b+c}{a}} = \sum_{cyc} \sqrt[2011]{\frac{R}{2r}.\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}}} \overset{0<\cos\frac{B-C}{2}\le 1}{\le}$$

$$\sqrt[2011]{\frac{R}{r}}.\sum_{cyc} \sqrt[2011]{\frac{1}{2\sin\frac{A}{2}}} \overset{A-G}{\le} \sqrt[2011]{\frac{R}{r}}.\sum_{cyc} \frac{\frac{1}{2\sin\frac{A}{2}} + 2010}{2011}$$

$$= \sqrt[2011]{\frac{R}{r}}.\left(\frac{6030}{2011} + \frac{1}{4022}.\sum_{cyc} \sqrt{\frac{bc(s-a)}{(s-a)(s-b)(s-c)}}\right) \overset{CBS}{\le}$$

$$\sqrt[2011]{\frac{R}{r}}.\left(\frac{6030}{2011} + \frac{1}{4022}.\frac{1}{r.\sqrt{s}}.\sqrt{s^2+4Rr+r^2}.\sqrt{s}\right) \overset{Gerretsen}{\le}$$

$$\sqrt[2011]{\frac{R}{r}}.\left(\frac{6030}{2011} + \frac{1}{4022}.\frac{1}{r}.\sqrt{4R^2+8Rr+4r^2}\right)$$

$$\therefore \boxed{\sum_{cyc} \sqrt[2011]{\frac{m_a(m_b+m_c)}{m_b m_c}} \le \sqrt[2011]{\frac{R}{r}}.\left(\frac{6031}{2011} + \frac{1}{2011}.\frac{R}{r}\right)} \to (2)$$

$\therefore (1), (2) \Rightarrow$ in order to prove :

$$\sum_{cyc}\sqrt[2011]{\frac{a(b+c)}{bc}}+\frac{R^2}{r^2}\ge 4+\sum_{cyc}\sqrt[2011]{\frac{m_a(m_b+m_c)}{m_bm_c}}\text{, it suffices to prove :}$$

$$3.\sqrt[6033]{\frac{16r}{R}+\frac{R^2}{r^2}}\ge 4+\sqrt[2011]{\frac{R}{r}.\left(\frac{6031}{2011}+\frac{1}{2011}.\frac{R}{r}\right)}$$

$$\Leftrightarrow \boxed{3.\sqrt[6033]{\frac{16}{t}+t^2}\overset{(*)}{\ge} 4+\sqrt[2011]{t}.\left(\frac{6031}{2011}+\frac{t}{2011}\right)}\left(t=\frac{R}{r}\overset{\text{Euler}}{\ge}2\right)$$

$$\text{Let } f(t)=3.\sqrt[6033]{\frac{16}{t}+t^2}-4-\sqrt[2011]{t}.\left(\frac{6031}{2011}+\frac{t}{2011}\right)\ \forall\, t\ge 2 \text{ and then :}$$

$$f'(t)=2t-\frac{\sqrt[2011]{t}}{2011}-\frac{\frac{t}{2011}+\frac{6031}{2011}}{2011.t^{\frac{2010}{2011}}}-\frac{\sqrt[6033]{16}}{2011.t^{\frac{6034}{6033}}}\text{ and}$$

$$f''(t)=2\left(1-\frac{1}{4044121.t^{\frac{2010}{2011}}}\right)+\frac{2010\left(\frac{t}{2011}+\frac{6031}{2011}\right)}{4044121.t^{\frac{4021}{2011}}}+\frac{6034.\sqrt[6033]{16}}{12132363.t^{\frac{12067}{6033}}}$$

$$\text{Now, }\because t\ge 2\therefore \frac{2010}{2011}.\ln t>-\ln 2\Rightarrow t^{\frac{2010}{2011}}>\frac{1}{2}\therefore 4044121.t^{\frac{2010}{2011}}>4044121.\frac{1}{2}>1$$

$$\Rightarrow 1-\frac{1}{4044121.t^{\frac{2010}{2011}}}>0\therefore f''(t)>0\ \forall\, t\ge 2\Rightarrow f'(t)\text{ is }\uparrow\text{ on }[2,\infty)$$

$$\Rightarrow f'(t)\ge f'(2)\approx 3.998508>0\ \forall\, t\ge 2\Rightarrow f(t)\ge f(2)=0$$

$$\Rightarrow 3.\sqrt[6033]{\frac{16}{t}+t^2}-4-\sqrt[2011]{t}.\left(\frac{6031}{2011}+\frac{t}{2011}\right)\ge 0\Rightarrow (*)\text{ is true}$$

$$\Rightarrow \frac{R^2}{r^2}-4\overset{(\bullet)}{\ge}\sum_{cyc}\sqrt[2011]{\frac{m_a(m_b+m_c)}{m_bm_c}}-\sum_{cyc}\sqrt[2011]{\frac{a(b+c)}{bc}}$$

$$\text{Let } F(n)=t^n-2^n\ \forall\, t=\frac{R}{r}\ge 2\ (t\to\text{fixed})\text{ and }\forall\, n\ge 2\text{ and then :}$$

$$F'(n)=t^n.(\ln t)-2^n.(\ln 2)\ge 0$$

$$(\because t^n\ge 2^n\text{ and }\ln t\ge \ln 2\Rightarrow t^n.(\ln t)-2^n.(\ln 2)\ge 0)$$

$$\therefore F(n)\text{ is }\uparrow\ \forall\, n\ge 2\Rightarrow\ F(n)\ge\ F(2)\Rightarrow\left(\frac{R}{r}\right)^n-2^n$$

$$\ge\frac{R^2}{r^2}-4\overset{\text{via }(\bullet)}{\ge}\sum_{cyc}\sqrt[2011]{\frac{m_a(m_b+m_c)}{m_bm_c}}-\sum_{cyc}\sqrt[2011]{\frac{a(b+c)}{bc}}$$

$$\therefore\sum_{cyc}\sqrt[2011]{\frac{a(b+c)}{bc}}+\frac{R^n}{r^n}\ge 2^n+\sum_{cyc}\sqrt[2011]{\frac{m_a(m_b+m_c)}{m_bm_c}}$$

$\forall\,\Delta\,ABC$ and $\forall\, n\ge 2$, " = " iff $\Delta\,ABC$ is equilateral (QED)