ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc} \frac{g_a^2}{w_b^2 + w_c^2} + \frac{R^3 - 8r^3}{2r^3} \geq \sum_{cyc} \frac{m_a^2}{h_b^2 + h_c^2}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, } \sum_{cyc} m_a^4 = \left(\sum_{cyc} m_a^2\right)^2 - 2\sum_{cyc} m_a^2 m_b^2$$

$$= \frac{9}{16}\left(\sum_{cyc} a^2\right)^2 - 2.\frac{9}{16}.\sum_{cyc} a^2b^2 = \frac{9}{16}.\sum_{cyc} a^4 = \frac{9}{16}.\left(2\sum_{cyc} a^2b^2 - 16r^2s^2\right)$$

$$\overset{\text{Goldstone}}{\leq} \frac{9}{16}.\left(8R^2s^2 - 16r^2s^2\right) \Rightarrow \sum_{cyc} m_a^4 \leq \frac{9s^2}{2}.\left(R^2 - 2r^2\right) \to (1)$$

$$w_a \leq w_b \Leftrightarrow \frac{2bc\cos\frac{A}{2}}{b+c} \leq \frac{2ca\cos\frac{B}{2}}{c+a} \Leftrightarrow \frac{4R\cos\frac{B}{2}\sin\frac{B}{2}}{\cos\frac{B}{2}}.\frac{1}{b+c} \leq \frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{\cos\frac{A}{2}}.\frac{1}{c+a}$$

$$\Leftrightarrow \frac{(b+c)^2(s-b)(s-c)}{bc} \geq \frac{(c+a)^2(s-c)(s-a)}{ca}$$

$$\Leftrightarrow a(b+c)^2(c+a-b) \geq b(c+a)^2(b+c-a)$$

$$\Leftrightarrow ab\left(a^2 - b^2\right) + c^3(a-b) + 3abc(a-b) + c^2\left(a^2 - b^2\right) \geq 0$$

$$\Leftrightarrow (a-b)\left(\left(ab + c^2\right)(a+b) + c^3 + 3abc\right) \geq 0 \to \text{true for } a \geq b$$

$$\Rightarrow w_a \leq w_b \text{ for } a \geq b \text{ and analogs} \to (2)$$

$$\text{Now, WLOG assuming } a \geq b \geq c \Rightarrow h_a^2 \leq h_b^2 \leq h_c^2 \text{ and}$$

$$\frac{1}{w_b^2 + w_c^2} \leq \frac{1}{w_c^2 + w_a^2} \leq \frac{1}{w_a^2 + w_b^2} \text{ (via (2))} \therefore \sum_{cyc} \frac{g_a^2}{w_b^2 + w_c^2} \geq \sum_{cyc} \frac{h_a^2}{w_b^2 + w_c^2}$$

$$\overset{\text{Chebyshev}}{\geq} \frac{1}{3}.\frac{\sum_{cyc} a^2b^2}{4R^2}.\sum_{cyc} \frac{1}{w_b^2 + w_c^2} \geq \frac{1}{36R^2}.\left(\sum_{cyc} ab\right)^2.\sum_{cyc} \frac{1}{s(s-b+s-c)}$$

$$\geq \frac{1}{36R^2}.3abc\left(\sum_{cyc} a\right).\frac{1}{4Rrs^2}.\left(\sum_{cyc} ab\right) \geq \frac{1}{36R^2}.24Rrs^2.\frac{1}{4Rrs^2}.18Rr$$

$$\left(\because s^2 - 14Rr + r^2 = s^2 - 16Rr + 5r^2 + 2r(R - 2r) \overset{\text{Gerretsen + Euler}}{\geq} 0\right)$$

$$\Rightarrow s^2 + 4Rr + r^2 \geq 18Rr$$

$$\therefore \boxed{\sum_{cyc} \frac{g_a^2}{w_b^2 + w_c^2} \geq \frac{3r}{R}} \to (3)$$

$$\text{Again}, \sum_{cyc} \frac{m_a^2}{h_b^2+h_c^2} \overset{CBS}{\le} \sqrt{\sum_{cyc} m_a^4} \cdot \sqrt{\sum_{cyc}\left(\frac{1}{h_b^2+h_c^2}\right)^2} \overset{A-G}{\le} \sqrt{\sum_{cyc} m_a^4} \cdot \sqrt{\sum_{cyc}\frac{h_a^2}{4h_a^2h_b^2h_c^2}}$$

$$\overset{\text{via }(1)}{\le} \sqrt{\frac{9s^2}{2}.(R^2-2r^2)} \cdot \sqrt{\frac{\sum_{cyc} a^2b^2}{16R^2.\frac{4r^4s^4}{R^2}}} \overset{\text{Goldstone}}{\le} \sqrt{\frac{9s^2}{2}.(R^2-2r^2)} \cdot \sqrt{\frac{4R^2s^2}{16R^2.\frac{4r^4s^4}{R^2}}}$$

$$\therefore \boxed{\sum_{cyc} \frac{m_a^2}{h_b^2+h_c^2} \le \sqrt{\frac{9R^2(R^2-2r^2)}{32r^4}}} \to (4) \therefore (3),(4) \Rightarrow \text{it suffices to prove :}$$

$$\frac{3r}{R} + \frac{R^3-8r^3}{2r^3} \ge \sqrt{\frac{9R^2(R^2-2r^2)}{32r^4}} \Leftrightarrow \left(\frac{R\left(R^3-8r^3\right)+6r^4}{2Rr^3}\right)^2 \ge \frac{9R^2\left(R^2-2r^2\right)}{32r^4}$$

$$\Leftrightarrow 8t^8 - 9t^6 - 128t^5 + 114t^4 + 512t^2 - 768t + 288 \ge 0 \ \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\left(8t^6+32t^5+87t^4+92t^3+134t^2+168t+648\right)+1152\right) \ge 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \therefore \sum_{cyc} \frac{g_a^2}{w_b^2+w_c^2} + \frac{R^3-8r^3}{2r^3} \ge \sum_{cyc} \frac{m_a^2}{h_b^2+h_c^2}$$

$\forall \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)