ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}}+\frac{R^3}{r^3}\geq 8+max\left\{\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}},\sum_{cyc}\sqrt[2023]{\frac{w_a^2}{w_b^2+w_c^2}}\right\}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}}\overset{Panaitopol}{\geq}\sum_{cyc}\sqrt[2023]{\frac{4r^2}{R^2}\cdot\frac{h_a^2}{h_b^2+h_c^2}}=\sum_{cyc}\sqrt[2023]{\frac{4r^2}{R^2}\cdot\frac{b^2c^2}{a^2(b^2+c^2)}}$$

$$\overset{Bandila}{\geq}\sum_{cyc}\sqrt[2023]{\frac{4r^2}{R^2}\cdot\frac{b^2c^2}{a^2.\frac{R}{r}.bc}}=\sqrt[2023]{\frac{4r^3}{R^3}}\cdot\sum_{cyc}\sqrt[2023]{\frac{bc}{a^2}}\overset{A-G}{\geq}3.\sqrt[2023]{\frac{4r^3}{R^3}}\cdot\sqrt[6069]{\prod_{cyc}\frac{bc}{a^2}}$$

$$\therefore\sum_{cyc}\sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}}\geq 3.\sqrt[2023]{\frac{4r^3}{R^3}}\rightarrow(1)$$

$$\text{Now, via Power}-\text{Mean Inequality},\left(\frac{\sum_{cyc}x^{\frac{1}{2023}}}{3}\right)^{2023}\leq\frac{\sum_{cyc}x}{3}$$

$$\Rightarrow\sum_{cyc}x^{\frac{1}{2023}}\leq 3.\sqrt[2023]{\frac{\sum_{cyc}x}{3}}\rightarrow(i)$$

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Now, }(b+c)^2\overset{?}{\geq}32Rr\cos^2\frac{A}{2}\overset{via\ (i)}{=}8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$=8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)}=4a(b+c-a)$$

$$\Leftrightarrow(b+c)^2+4a^2-4a(b+c)\overset{?}{\geq}0$$

$$\Leftrightarrow(b+c-2a)^2\overset{?}{\geq}0\rightarrow\text{true}\therefore b+c\geq\sqrt{32Rr}.\cos\frac{A}{2}\text{ and analogs}\rightarrow(ii)$$

$$\text{We have : }\frac{w_a}{h_a}=\frac{\frac{2bc}{b+c}\cos\frac{A}{2}.a}{2rs}\overset{via\ (ii)}{\leq}\frac{2.4Rrs.\cos\frac{A}{2}}{2rs.4.\sqrt{2Rr}.\cos\frac{A}{2}}\Rightarrow w_a^2\leq\frac{R}{2r}.h_a^2\text{ and analogs}$$

$$\Rightarrow\sum_{cyc}\sqrt[2023]{\frac{w_a^2}{w_b^2+w_c^2}}\overset{via\ (i)}{\leq}3.\sqrt[2023]{\frac{\sum_{cyc}\frac{w_a^2}{w_b^2+w_c^2}}{3}}\overset{Reverse\ Bergstrom}{\leq}3.\sqrt[2023]{\frac{1}{12}\sum_{cyc}\left(\frac{w_a^2}{w_b^2}+\frac{w_a^2}{w_c^2}\right)}$$

$$\leq 3.\sqrt[2023]{\frac{1}{12}\cdot\frac{R}{2r}\cdot\sum_{cyc}\left(\frac{h_a^2}{h_b^2}+\frac{h_a^2}{h_c^2}\right)}=3.\sqrt[2023]{\frac{R}{24r}\sum_{cyc}\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)}$$

$$\overset{Bandila}{\leq} 3.\sqrt[2023]{\frac{R}{24r}\sum_{cyc}\left(\frac{R^2}{r^2}-2\right)} \therefore \sum_{cyc}\sqrt[2023]{\frac{w_a^2}{w_b^2+w_c^2}} \leq 3.\sqrt[2023]{\frac{R}{8r}\left(\frac{R^2}{r^2}-2\right)} \to (2)$$

$$\text{Also}, \sum_{cyc}\sqrt[2023]{\frac{a}{b+c}} \overset{via\ (ii)}{\leq} \sum_{cyc}\sqrt[2023]{\frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{4.\sqrt{2Rr}.\cos\frac{A}{2}}} = \sqrt[4046]{\frac{R}{2r}}.\sum_{cyc}\sqrt[2023]{\sin\frac{A}{2}}$$

$$\overset{Jensen}{\leq} 3.\sqrt[4046]{\frac{R}{2r}}.\sqrt[2023]{\frac{1}{2}}$$

$$\left(\therefore f(x) = \sqrt[2023]{\sin\frac{x}{2}}\ \forall x \in (0,\pi) \Rightarrow f''(x) = \frac{2022+\sin^2\frac{x}{2}}{16370116\left(\sin\frac{x}{2}\right)^{\frac{4045}{2023}}} < 0\right)$$

$$\therefore \sum_{cyc}\sqrt[2023]{\frac{a}{b+c}} \leq 3.\sqrt[2023]{\sqrt{\frac{R}{2r}}.\frac{1}{2}} \to (3) \therefore (1)\ and\ (2) \Rightarrow \text{in order to prove}:$$

$$\sum_{cyc}\sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}} + \frac{R^3}{r^3} - 8 \geq \sum_{cyc}\sqrt[2023]{\frac{w_a^2}{w_b^2+w_c^2}}, \text{it suffices to prove}:$$

$$3.\sqrt[2023]{\frac{4r^3}{R^3}} + \frac{R^3}{r^3} - 8 \geq 3.\sqrt[2023]{\frac{R}{8r}\left(\frac{R^2}{r^2}-2\right)}$$

$$\Leftrightarrow \boxed{t^3 - 8 + 3.\sqrt[2023]{\frac{4}{t^3}} - 3.\sqrt[2023]{\frac{t}{8}(t^2-2)} \overset{(*)}{\geq} 0\ \left(t = \frac{R}{r}\right)}$$

$$\text{Let } F(t) = t^3 - 8 + 3.\sqrt[2023]{\frac{4}{t^3}} - 3.\sqrt[2023]{\frac{t}{8}(t^2-2)}\ \forall\, t \geq 2\ \text{and then}:$$

$$F'(t) = 3t^2 - \frac{6t^{\frac{2024}{2023}}}{2023.\sqrt[2023]{8}.(t^2-2)^{\frac{2022}{2023}}} - \frac{3.\sqrt[2023]{t^2-2}}{2023.\sqrt[2023]{8}.t^{\frac{2022}{2023}}} - \frac{9.\sqrt[2023]{4}}{2023t^{\frac{2026}{2023}}} \to (?)$$

$$\text{Now}, t^2(t^2-2)^{\frac{2022}{2023}}.2023.\sqrt[2023]{8} \overset{Euler}{\geq} t^2.2^{\frac{2022}{2023}}.2023.2^{\frac{3}{2023}}$$

$$= t^2.\left(2023.2^{\frac{2025}{2023}}\right) \overset{t\geq 2}{>} t^{\frac{2024}{2023}}.(6) \Rightarrow t^2 > \frac{6t^{\frac{2024}{2023}}}{2023.\sqrt[2023]{8}.(t^2-2)^{\frac{2022}{2023}}} \to (\bullet)$$

$$\text{Again}, t^2.2023.\sqrt[2023]{8}.t^{\frac{2022}{2023}} = t^{\frac{6068}{2023}}.\left(2023.\sqrt[2023]{8}\right) > \sqrt[2023]{t^2-2}.(3)$$

$$\left(\because t^{\frac{6068}{2023}} \overset{t\geq 2}{>} t^{\frac{2}{2023}} \Rightarrow t^{\frac{6068}{2023}} > \sqrt[2023]{t^2-2}\ and\ 2023^{2023}.8 > 3^{2023}\right.$$

$$\left.\Rightarrow 2023.\sqrt[2023]{8} > 3\right)$$

$$\Rightarrow t^2 > \frac{3.\sqrt[2023]{t^2-2}}{2023.\sqrt[2023]{8}.t^{\frac{2022}{2023}}} \to (\bullet\bullet)$$

$$Also, t^2.2023t^{\frac{2026}{2023}} \overset{t\geq 2}{>} 2023 > 9.\sqrt[2023]{4} \therefore t^2 > \frac{9.\sqrt[2023]{4}}{2023t^{\frac{2026}{2023}}} \to (\bullet\bullet\bullet)$$

$\therefore (\bullet) + (\bullet\bullet) + (\bullet\bullet\bullet) \Rightarrow$ LHS of (?) is true $\Rightarrow F'(t) > 0\ \forall\, t \geq 2 \Rightarrow F(t)$ is $\uparrow$ on $[2,\infty)$ $\Rightarrow F(t) \geq F(2) = 0 \Rightarrow (*)$ is true

$$\therefore \boxed{\sum_{cyc} \sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}} + \frac{R^3}{r^3} - 8 \overset{(\blacksquare)}{\geq} \sum_{cyc} \sqrt[2023]{\frac{w_a^2}{w_b^2+w_c^2}}}$$

Again, (1) and (3) $\Rightarrow$ in order to prove : $\sum_{cyc} \sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}} + \frac{R^3}{r^3} - 8$

$\geq \sum_{cyc} \sqrt[2023]{\frac{a}{b+c}}$, it suffices to prove : $3.\sqrt[2023]{\frac{4r^3}{R^3}} + \frac{R^3}{r^3} - 8 \geq 3.\sqrt[2023]{\sqrt{\frac{R}{2r}}.\frac{1}{2}}$

$$\Leftrightarrow \boxed{t^3 - 8 + 3.\sqrt[2023]{\frac{4}{t^3}} - 3.\sqrt[2023]{\sqrt{t}.\frac{1}{2}} \overset{(**)}{\geq} 0}$$

Let $P(t) = t^3 - 8 + 3.\sqrt[2023]{\frac{4}{t^3}} - 3.\sqrt[2023]{\sqrt{t}.\frac{1}{2}}\ \forall\, t \geq 2$ and then :

$$P'(t) = 3t^2 - \frac{3}{2023.2^{\frac{4049}{4046}}.t^{\frac{4045}{4046}}} - \frac{9.\sqrt[2023]{4}}{2023.t^{\frac{2026}{2023}}} \to (??)$$

$$Now, t^2.2023.2^{\frac{4049}{4046}}.t^{\frac{4045}{4046}} = t^{2+\frac{4045}{4046}}.\left(2023.2^{\frac{4049}{4046}}\right) \overset{t\geq 2}{>} 2023.2^{\frac{4049}{4046}} > 3$$

$$\Rightarrow t^2 > \frac{3}{2023.2^{\frac{4049}{4046}}.t^{\frac{4045}{4046}}} \to (\bullet\bullet\bullet\bullet)$$

$$Also, t^2.2023.t^{\frac{2026}{2023}} = t^{2+\frac{2026}{2023}}.(2023) \overset{t\geq 2}{>} 2023 > 9.\sqrt[2023]{4} \Rightarrow t^2 > \frac{9.\sqrt[2023]{4}}{2023.t^{\frac{2026}{2023}}}$$

$\to (\bullet\bullet\bullet\bullet\bullet) \therefore (\bullet\bullet\bullet\bullet) + (\bullet\bullet\bullet\bullet\bullet) \Rightarrow$ LHS of (??) is true $\Rightarrow P'(t) > 0\ \forall\, t \geq 2$ $\Rightarrow P(t)$ is $\uparrow$ on $[2,\infty) \Rightarrow P(t) \geq F(2) = 0 \Rightarrow (**)$ is true

$$\therefore \boxed{\sum_{cyc} \sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}} + \frac{R^3}{r^3} - 8 \overset{(\blacksquare\blacksquare)}{\geq} \sum_{cyc} \sqrt[2023]{\frac{a}{b+c}}} \therefore (\blacksquare)(\blacksquare\blacksquare) \Rightarrow$$

$$\sum_{cyc} \sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}} + \frac{R^3}{r^3} - 8 \geq max\left\{\sum_{cyc} \sqrt[2023]{\frac{a}{b+c}}, \sum_{cyc} \sqrt[2023]{\frac{w_a^2}{w_b^2+w_c^2}}\right\}$$

$$\Rightarrow \sum_{cyc} \sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}} + \frac{R^3}{r^3} \geq 8 + max\left\{\sum_{cyc} \sqrt[2023]{\frac{a}{b+c}}, \sum_{cyc} \sqrt[2023]{\frac{w_a^2}{w_b^2+w_c^2}}\right\}$$

$\forall\, \Delta\, ABC$, " = " iff Δ ABC is equilateral (QED)