ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta\,ABC$, the following relationship holds :

$$\frac{R}{2r}\sum_{cyc}\frac{a^{2024}}{b^{2024}+c^{2024}}+\frac{R^{2025}}{r^{2025}}\geq 2^{2025}+\sum_{cyc}\frac{a^2}{b^2+c^2}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } F(n)=t^n-2^n\ \forall\, t=\frac{R}{r}\geq 2\ (t\to \text{fixed}) \text{ and } \forall\, n\geq 2 \text{ and then :}$$

$$F'(n)=t^n.(\ln t)-2^n.(\ln 2)\geq 0\ \begin{pmatrix}\because t^n\geq 2^n \text{ and } \ln t\geq \ln 2\\ \Rightarrow t^n.(\ln t)-2^n.(\ln 2)\geq 0\end{pmatrix}$$

$$\therefore F(n) \text{ is } \uparrow\ \forall\, n\geq 2\Rightarrow F(n)\geq F(2)\Rightarrow \text{when both } t=\frac{R}{r} \text{ and } n \text{ vary}, \left(\frac{R}{r}\right)^n-2^n\geq$$

$$\frac{R^2}{r^2}-4\Rightarrow \frac{R}{2r}\sum_{cyc}\frac{a^{2024}}{b^{2024}+c^{2024}}+\frac{R^{2025}}{r^{2025}}-2^{2025}\overset{\text{Nesbitt}}{\geq}\frac{R}{2r}.\frac{3}{2}+\frac{R^2}{r^2}-4\to(1)$$

$$\text{Also, }\sum_{cyc}\frac{a^2}{b^2+c^2}=\frac{1}{4}\sum_{cyc}\frac{4a^2}{b^2+c^2}\overset{\text{Reverse Bergstrom}}{\leq}\frac{1}{4}\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{a^2}{c^2}\right)$$

$$=\frac{1}{4}\sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)^2-2\right)\overset{\text{Bandila}}{\leq}\frac{1}{4}\sum_{cyc}\left(\left(\frac{R}{r}\right)^2-2\right)\Rightarrow\sum_{cyc}\frac{a^2}{b^2+c^2}\leq\frac{3R^2}{4r^2}-\frac{3}{2}\to(2)$$

$$\therefore \text{via } (1) \text{ and } (2), \text{in order to prove : }\frac{R}{2r}\sum_{cyc}\frac{a^{2024}}{b^{2024}+c^{2024}}+\frac{R^{2025}}{r^{2025}}-2^{2025}\geq$$

$$\sum_{cyc}\frac{a^2}{b^2+c^2}, \text{it suffices to prove : }\frac{R}{2r}.\frac{3}{2}+\frac{R^2}{r^2}-4\geq\frac{3R^2}{4r^2}-\frac{3}{2}\Leftrightarrow$$

$$\frac{4R^2-16r^2+3Rr}{4r^2}\geq\frac{3R^2-6r^2}{4r^2}\Leftrightarrow R^2+3Rr-10r^2\geq 0\Leftrightarrow (R-2r)(R+5r)\geq 0$$

$$\to \text{true via Euler }\therefore \frac{R}{2r}\sum_{cyc}\frac{a^{2024}}{b^{2024}+c^{2024}}+\frac{R^{2025}}{r^{2025}}-2^{2025}\geq\sum_{cyc}\frac{a^2}{b^2+c^2}$$

$$\forall\,\Delta\,ABC, ''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$