ROMANIAN MATHEMATICAL MAGAZINE

**In any $\Delta$ ABC, the following relationship holds :**

$$\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{3R^2}{4r^2}\ge\sum_{cyc}\frac{a}{b}+9g_ag_bg_c$$

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$$\text{Triangle inequality}\Rightarrow g_a\le AI+r\overset{?}{\le}w_a\Leftrightarrow\frac{r}{\sin\frac{A}{2}}+r\overset{?}{\le}\frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow\frac{r}{\sin\frac{A}{2}}+r\overset{?}{\le}\frac{8Rrs\cos\frac{A}{2}}{4R(b+c)\sin\frac{A}{2}\cos\frac{A}{2}}\Leftrightarrow\frac{1}{\sin\frac{A}{2}}+1\overset{?}{\le}\frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow\frac{1}{\sin\frac{A}{2}}+1\overset{?}{\le}+\frac{a}{(b+c)\sin\frac{A}{2}}+\frac{1}{\sin\frac{A}{2}}\Leftrightarrow(b+c)\sin\frac{A}{2}\overset{?}{\le}a$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2}\overset{?}{\le}4R\sin\frac{A}{2}\cos\frac{A}{2}\Leftrightarrow\cos\frac{B-C}{2}\overset{?}{\le}1\to\text{true}$$

$$\therefore g_a\le w_a\overset{A-G}{\le}\sqrt{s(s-a)}\text{ and analogs}$$

$$\Rightarrow 9g_ag_bg_c\le\sqrt{s(s-a)}.\sqrt{s(s-b)}.\sqrt{s(s-c)}\therefore 9g_ag_bg_c\overset{(\bullet)}{\le}9rs^2$$

$$\text{Now, }\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{3R^2}{4r^2}\ge\sum_{cyc}\frac{a}{b}+9g_ag_bg_c$$

$$\Leftrightarrow\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{3R^2}{4r^2}\ge\sum_{cyc}\frac{a+b+c-(b+c)}{b}+9g_ag_bg_c$$

$$\Leftrightarrow\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{3R^2}{4r^2}+3+\sum_{cyc}\frac{c}{b}\ge\frac{2s}{4Rrs}\sum_{cyc}ab+9g_ag_bg_c$$

$$\Leftrightarrow\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{3R^2}{4r^2}+3+\sum_{cyc}\frac{c}{b}\overset{(*)}{\ge}\frac{1}{2Rr}\sum_{cyc}ab+9g_ag_bg_c$$

$$\text{We have : }\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}\ge\sum_{cyc}\frac{s(s-a)\sqrt{bc}\cos\frac{A}{2}\tan\frac{A}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}\overset{G-H}{\ge}$$

$$\sum_{cyc}\frac{s(s-a).\frac{2bc}{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}.\sin\frac{A}{2}}{\frac{r}{4R}.\frac{4R}{s}}\underset{\because s\tan\frac{A}{2}=r_a}{\overset{0<\cos\frac{B-C}{2}\le 1\text{ and}}{\ge}}\sum_{cyc}\frac{s(s-a)h_ar_a}{r}\overset{A-G}{\ge}\frac{3s}{r}.\sqrt[3]{r^2s.\frac{2r^2s^2}{R}.rs^2}\overset{?}{\ge}9rs^2$$

$$\Leftrightarrow\frac{r^2s.\frac{2r^2s^2}{R}.rs^2}{r^3}\overset{?}{\ge}27r^3s^3\Leftrightarrow 2s^2\overset{?}{\ge}27Rr\to\text{true}\because 2s^2\overset{\text{Gerretsen}}{\ge}$$

$$27Rr+5r(R-2r)\overset{\text{Euler}}{\geq}27Rr\therefore\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}\geq 9rs^2\overset{\text{via }(\bullet)}{\geq}9g_ag_bg_c\to(1)$$

$$\text{Again,}\frac{3R^2}{4r^2}+3+\sum_{cyc}\frac{c}{b}-\frac{1}{2Rr}\sum_{cyc}ab\overset{\substack{A-G\\ and\\ \text{Gerretsen}}}{\geq}\frac{3R^2}{4r^2}+6-\frac{4R^2+8Rr+4r^2}{2Rr}$$

$$=\frac{3R^2+24r^2}{4r^2}-\frac{4R^2+8Rr+4r^2}{2Rr}\overset{?}{\geq}0\Leftrightarrow 3t^3-8t^2+8t-8\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(2t^2+t(t-2)+4\right)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\therefore\frac{3R^2}{4r^2}+3+\sum_{cyc}\frac{c}{b}\geq\frac{1}{2Rr}\sum_{cyc}ab\to(2)$$

$$\therefore(1)+(2)\Rightarrow(*)\text{ is true}\therefore\sum_{cyc}\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{3R^2}{4r^2}\geq\sum_{cyc}\frac{a}{b}+9g_ag_bg_c\ \forall\Delta ABC,$$

$''=''$ iff $\Delta$ ABC is equilateral (QED)