ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC, the following relationship holds :

$$\sqrt[3]{\frac{2}{3}\cdot\sum_{cyc}\frac{a^3}{b^3+c^3}}+\frac{R^3-8r^3}{r^3}\geq\frac{2}{3}\cdot\sum_{cyc}\frac{a^4}{b^4+c^4}$$

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$$\frac{2}{3}\cdot\sum_{cyc}\frac{a^4}{b^4+c^4}\overset{A-G}{\leq}\frac{2}{3}\cdot\sum_{cyc}\frac{a^4}{bc(b^2+c^2)}=\frac{2}{3}\cdot\sum_{cyc}\frac{a^2\left(\sum_{cyc}a^2-(b^2+c^2)\right)}{bc(b^2+c^2)}$$

$$=\frac{2}{3}\cdot\left(\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{a^2}{bc(b^2+c^2)}\right)-\frac{1}{4Rrs}\cdot\sum_{cyc}a^3\right)$$

$$\overset{A-G}{\leq}\frac{2}{3}\cdot\left((s^2-4Rr-r^2)\left(\sum_{cyc}\frac{a^2}{b^2c^2}\right)-\frac{2s(s^2-6Rr-3r^2)}{4Rrs}\right)$$

$$=\frac{2}{3}\cdot\left(\left(\frac{s^2-4Rr-r^2}{16R^2r^2s^2}\right)\left(2\sum_{cyc}a^2b^2-16r^2s^2\right)-\frac{s^2-6Rr-3r^2}{2Rr}\right)$$

$$\overset{Goldstone}{\leq}\frac{2}{3}\cdot\left(\left(\frac{s^2-4Rr-r^2}{16R^2r^2s^2}\right)(8R^2s^2-16r^2s^2)-\frac{s^2-6Rr-3r^2}{2Rr}\right)$$

$$=\frac{1}{3}\cdot\left(\frac{(s^2-4Rr-r^2)(R^2-2r^2)-Rr(s^2-6Rr-3r^2)}{R^2r^2}\right)$$

$$=\frac{(R^2-Rr-2r^2)s^2-(4Rr+r^2)(R^2-2r^2)+Rr(6Rr+3r^2)}{3R^2r^2}$$

$$\overset{Gerretsen}{\leq}\frac{(R^2-Rr-2r^2)(4R^2+4Rr+3r^2)-(4Rr+r^2)(R^2-2r^2)+Rr(6Rr+3r^2)}{3R^2r^2}$$

$$\left(\because R^2-Rr-2r^2=(R-2r)(R+r)\overset{Euler}{\geq}0\right)\overset{?}{\leq}\frac{R^3-7r^3}{r^3}$$

$$\Leftrightarrow 3t^5-4t^4+4t^3-17t^2+4\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(3t^4+2t^3+7t^2+\frac{t(t-2)+t^2-4}{2}\right)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{Euler}{\geq}2$$

$$\therefore\frac{2}{3}\cdot\sum_{cyc}\frac{a^4}{b^4+c^4}\leq\frac{R^3-7r^3}{r^3}\rightarrow(1)$$

Now, $\sqrt[3]{\frac{2}{3}\cdot\sum_{cyc}\frac{a^3}{b^3+c^3}}\overset{Nesbitt}{\geq}\sqrt[3]{\frac{2}{3}\cdot\frac{3}{2}}=1\ \therefore\sqrt[3]{\frac{2}{3}\cdot\sum_{cyc}\frac{a^3}{b^3+c^3}}+\frac{R^3-8r^3}{r^3}\geq\frac{R^3-7r^3}{r^3}$

$$\overset{via\ (1)}{\geq}\frac{2}{3}\cdot\sum_{cyc}\frac{a^4}{b^4+c^4}\ \forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$