ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\sqrt[3]{\frac{a}{b+c}}+\frac{R^{2024}}{r^{2024}}\ge 2^{2024}+\sum_{cyc}\sqrt[3]{\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}}$$

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$$\frac{\sum_{cyc}a^2}{\sum_{cyc}ab}\overset{?}{\le}\frac{R}{2r}\Leftrightarrow R(s^2+4Rr+r^2)\overset{?}{\ge}4r(s^2-4Rr-r^2)$$

$$\Leftrightarrow (R-4r)s^2+(R+4r)(4Rr+r^2)\underset{(*)}{\overset{?}{\ge}}0$$

$$\text{Now, LHS of }(*)=(R-2r)s^2-2rs^2+(R+4r)(4Rr+r^2)\overset{\text{Gerretsen}}{\ge}$$

$$(R-2r)(16Rr-5r^2)-2r(4R^2+4Rr+3r^2)+(R+4r)(4Rr+r^2)\overset{?}{\ge}0$$

$$\Leftrightarrow 3R^2-7Rr+2r^2\overset{?}{\ge}0\Leftrightarrow(R-2r)(3R-r)\overset{?}{\ge}0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r$$

$$\therefore\frac{\sum_{cyc}a^2}{\sum_{cyc}ab}\le\frac{R}{2r}\to(1)$$

$$\text{Now, }\sum_{cyc}\sqrt[3]{\frac{a}{b+c}}=\frac{1}{\sqrt[3]{2}}\sum_{cyc}\sqrt[3]{\frac{2a}{b+c}.1.1}\overset{G-H}{\ge}\frac{3}{\sqrt[3]{2}}\sum_{cyc}\frac{\frac{2a}{b+c}}{\frac{2a}{b+c}+\frac{2a}{b+c}+1}$$

$$=\frac{6}{\sqrt[3]{2}}\sum_{cyc}\frac{a}{4a+b+c}=\frac{6}{\sqrt[3]{2}}\sum_{cyc}\frac{a^2}{4a^2+ab+ca}\overset{\text{Bergstrom}}{\ge}\frac{6}{\sqrt[3]{2}}.\frac{\left(\sum_{cyc}a\right)^2}{4\sum_{cyc}a^2+2\sum_{cyc}ab}$$

$$=\frac{6}{\sqrt[3]{2}}.\frac{u+2v}{4u+2v}\left(u=\sum_{cyc}a^2,v=\sum_{cyc}ab\right)\overset{?}{\ge}\frac{3}{\sqrt[3]{2}}.\frac{v}{u}$$

$$\Leftrightarrow 2u^2+4uv\overset{?}{\ge}4uv+2v^2\Leftrightarrow u\overset{?}{\ge}v\to\text{true}\therefore\sum_{cyc}\sqrt[3]{\frac{a}{b+c}}\ge\frac{3}{\sqrt[3]{2}}.\frac{\sum_{cyc}ab}{\sum_{cyc}a^2}$$

$$\overset{\text{via }(1)}{\ge}\frac{3}{\sqrt[3]{2}}.\frac{2r}{R}\therefore\sum_{cyc}\sqrt[3]{\frac{a}{b+c}}\ge\frac{3}{\sqrt[3]{2}}.\frac{2r}{R}\to(2)$$

$$\text{We have : }r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Again, }\sum_{cyc}\sqrt[3]{\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}}=\frac{1}{\sqrt[3]{2}}\sum_{cyc}\sqrt[3]{\frac{2r_a}{r_b+r_c}.1.1}\overset{A-G}{\le}\frac{1}{3.\sqrt[3]{2}}\sum_{cyc}\left(\frac{2r_a}{r_b+r_c}+2\right)$$

$$=\frac{2}{3.\sqrt[3]{2}}.\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}\frac{1}{r_b+r_c}\right)\overset{\text{via }(i)}{=}\frac{2}{3.\sqrt[3]{2}}.(4R+r).\sum_{cyc}\frac{1}{4R\cos^2\frac{A}{2}}$$

$$= \frac{2}{3.\sqrt[3]{2}}.\left(\frac{4R+r}{4R}\right).\left(1+\frac{(4R+r)^2}{s^2}\right) \overset{\substack{\text{Euler}\\ and\\ \text{Mitrinovic}}}{\leq} \frac{2}{3.\sqrt[3]{2}}.\left(\frac{\frac{9R}{2}}{4R}\right).\left(1+\frac{\frac{81R^2}{4}}{27r^2}\right)$$

$$\Rightarrow \sum_{cyc}\sqrt[3]{\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}} \leq \frac{3}{4.\sqrt[3]{2}}.\left(1+\frac{3R^2}{4r^2}\right) \to (3)$$

Let $F(n) = t^n - 2^n \ \forall\, t = \frac{R}{r} \overset{\text{Euler}}{\geq} 2$ ($t \to$ fixed) *and* $\forall\, n \geq 2$ *and* then :

$$F'(n) = t^n.(\ln t) - 2^n.(\ln 2) \geq 0$$

$$(\because t^n \geq 2^n \text{ and } \ln t \geq \ln 2 \Rightarrow t^n.(\ln t) - 2^n.(\ln 2) \geq 0)$$

$\therefore$ $F(n)$ is $\uparrow \forall\, n \geq 2 \Rightarrow F(n) \geq F(2) \Rightarrow$ when both t *and* n *vary*, $\left(\frac{R}{r}\right)^n - 2^n$

$$\geq \frac{R^2}{r^2} - 4 \Rightarrow \frac{R^{2024}}{r^{2024}} - 2^{2024} \geq \frac{R^2}{r^2} - 4 \to (4)$$

$$\text{So, } \sum_{cyc}\sqrt[3]{\frac{a}{b+c}} + \frac{R^{2024}}{r^{2024}} - 2^{2024} - \sum_{cyc}\sqrt[3]{\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}} \overset{\text{via (2),(3) and (4)}}{\geq}$$

$$\frac{3}{\sqrt[3]{2}}.\frac{2r}{R} + \frac{R^2}{r^2} - 4 - \frac{3}{4.\sqrt[3]{2}}.\left(1+\frac{3R^2}{4r^2}\right) = \frac{R^2-4r^2}{r^2} - \frac{3}{\sqrt[3]{2}}.\left(\frac{1}{4}+\frac{3R^2}{16r^2}-\frac{2r}{R}\right)$$

$$= \frac{R^2-4r^2}{r^2} - \frac{3}{16.\sqrt[3]{2}}.\left(\frac{3R^3+4Rr^2-32r^3}{Rr^2}\right) \geq$$

$$\frac{R^2-4r^2}{r^2} - \frac{1}{6}.\left(\frac{3R^3+4Rr^2-32r^3}{Rr^2}\right)$$

$$\left(\because \frac{3}{16.\sqrt[3]{2}} \approx 0.1488 < \frac{1}{6} \text{ and } 3R^3+4Rr^2-32r^3 \overset{\text{Euler}}{\geq} 24r^3+8r^3-32r^3 = 0\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow 3t^3 - 28t + 32 \overset{?}{\geq} 0 \Leftrightarrow (t-2)\big((t-2)(3t+12)+8\big) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow \sum_{cyc}\sqrt[3]{\frac{a}{b+c}} + \frac{R^{2024}}{r^{2024}} - 2^{2024} - \sum_{cyc}\sqrt[3]{\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}} \geq 0$$

$$\therefore \sum_{cyc}\sqrt[3]{\frac{a}{b+c}} + \frac{R^{2024}}{r^{2024}} \geq 2^{2024} + \sum_{cyc}\sqrt[3]{\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}}$$

$\forall\, \Delta\, ABC$, " = " iff $\Delta\, ABC$ is equi*lateral* (QED)