ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC holds:

$$\prod \sqrt[2024]{\frac{a}{b+c}}+\frac{R^{2024}}{r^{2024}} \geq \prod \sqrt[2024]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}}+2^{2024}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\prod \sqrt[2024]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}} \overset{AM-GM}{\leq} \sqrt[2024]{\frac{\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}}{8\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}}} = \frac{1}{2^{\frac{3}{2024}}} = \frac{1}{2^{\frac{3}{m}}}, (m = 2024),$$

$$\prod \sqrt[2024]{\frac{a}{b+c}} = \left(\frac{abc}{(a+b)(b+c)(c+a)}\right)^{\frac{1}{2024}} =$$

$$= \left(\frac{2Rr}{s^2+r^2+2Rr}\right)^{\frac{1}{2024}} \overset{Gerretsen}{\geq} \left(\frac{2Rr}{4R^2+6Rr+4r^2}\right)^{\frac{1}{2024}} \overset{Euler}{\geq}$$

$$\geq \left(\frac{1}{2}\frac{r^2}{R^2}\right)^{\frac{1}{2024}} \overset{Euler}{\geq} \left(\frac{r}{R}\right)^{\frac{3}{2024}} = \left(\frac{r}{R}\right)^{\frac{3}{m}}$$

$$\textit{Now we need to show } \left(\frac{r}{R}\right)^{\frac{3}{m}} + \left(\frac{R}{r}\right)^{m} \geq \frac{1}{2^{\frac{3}{m}}} + 2^m \textit{ or}$$

$$(2x)^{\frac{3}{m}}(x^m - 2^m) - \left(x^{\frac{3}{m}} - 2^{\frac{3}{m}}\right) \overset{\frac{R}{r}=x\geq 2}{\geq} 0$$

$$.\textit{let } f(x) = (2x)^{\frac{3}{m}}(x^m - 2^m) - \left(x^{\frac{3}{m}} - 2^{\frac{3}{m}}\right) \textit{ and}$$

$$f'(x) = 2^{\frac{3}{m}}\frac{3}{m}\, x^{\frac{3}{m}-1}(x^m - 2^m) + (2x)^{\frac{3}{m}}\, m x^{m-1} - \frac{3}{m}x^{\frac{3}{m}-1} \textit{ or,}$$

$$f'(x) = x^{\frac{3}{m}-1}\left(2^{\frac{3}{m}}\, m\, x^m - \frac{3}{m}\right) + \frac{3}{m}x^{\frac{3}{m}-1}\left(2^{\frac{3}{m}}x^m - 2^{\frac{3}{m}+m}\right) > 0 \textit{ as } x \geq 2.$$

$$f(2) = 0 \textit{ so } f(x) \geq f(2) = 0 \textit{ hence } (2x)^{\frac{3}{m}}(x^m - 2^m) - \left(x^{\frac{3}{m}} - 2^{\frac{3}{m}}\right) \geq 0 \ (\textit{proved})$$