ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{m_a}{h_b+h_c}+\sum \frac{h_a}{w_a+w_b}+\frac{m_a m_b m_c}{w_a w_b w_c}\frac{R^{2024}}{r^{2024}}\ge 2^{2024}+2\sum \frac{m_a^2}{w_b^2+w_c^2}$$

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Known:

$$\sum \frac{1}{h_a}=\frac{1}{r},\prod h_a \overset{Gm-Hm}{\ge} \left(\frac{1}{\sum \frac{1}{h_a}}\right)^3=27r^3 \text{ and}$$

$$\sum w_a \le \sum m_a \overset{Leuenberger}{\le} (4R+r)\le \frac{9R}{2}(Euler),$$

$$\frac{m_a m_b m_c}{w_a w_b w_c}\ge 1,\left[\sum \frac{1}{b^2+c^2}\right]\ge \frac{9}{2\sum a^2}(Bergstrom)\ge \frac{1}{2R^2}(Leibniz)$$

$$2\sum \frac{m_a}{h_b+h_c}\overset{m_a\ge h_a}{\ge}\sum \frac{h_a}{h_b+h_c}\overset{Nesbitt}{\ge}\frac{3}{2}\ (1),$$

$$\sum \frac{h_a}{w_a+w_b}\overset{AM-GM}{\ge} 3\left(\frac{\prod h_a}{\prod(w_a+w_b)}\right)^{\frac{1}{3}}\overset{AM-GM}{\ge} 3(27r^3)^{\frac{1}{3}}\frac{1}{\frac{2\sum w_a}{3}}\ge \frac{9r}{3R}=\frac{3r}{R} \text{ and}$$

$$2\sum \frac{m_a^2}{w_b^2+w_c^2}\le 2\sum \frac{m_a^2}{h_b^2+h_c^2}=\frac{2(4R^2)}{4}\sum \frac{2(b^2+c^2)-a^2}{a^2c^2+a^2b^2}=$$

$$=2R^2\left[\sum \frac{2}{a^2}-\sum \frac{1}{b^2+c^2}\right]\le 2R^2\left(\frac{2}{4r^2}-\frac{1}{4R^2}\right)(Steining)$$

$$Let\ \frac{R}{r}=x\ge 2, we\ need\ to\ show$$

$$\frac{3}{2}+\frac{3}{x}+x^{2024}\ge 2^{2024}+x^2-1\ or$$

$$2x^{2025}-2x^3-x\left(2^{2025}-5\right)+6\ge 0$$

$$Let\ f(x)=2x^{2025}-2x^3-x\left(2^{2025}-5\right)+6$$

$$f'(x)=2.(2025x^{2024})-6x^2-\left(2^{2025}-5\right)=$$

$$=(2025x^{2024}-6x^2)+\left(2025x^{2024}-2^{2025}\right)+5\ge 0$$

$$true\ as\ x\ge 2, f(x)\ is\ increasing.$$

$$Equality\ for\ x=2\ and$$

$$f(x)\ge f(2) or\ 2x^{2025}-2x^3-x\left(2^{2025}-5\right)+6\ge 0$$