ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum\frac{m_a}{h_b+h_c}+\sum\frac{h_a}{m_b+m_c}+\frac{h_ah_bh_c}{w_aw_bw_c}\frac{R^{2025}}{r^{2025}}\geq 2^{2025}+2\sum\frac{m_a^2}{m_b^2+m_c^2}$$

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$$\sum\frac{1}{h_a}=\frac{1}{r},\prod h_a\overset{Gm-Hm}{\geq}\left(\frac{1}{\sum\frac{1}{h_a}}\right)^3=27r^3\ and$$

$$\sum m_a\overset{Leunberger}{\leq}(4R+r)\leq\frac{9R}{2}(Euler),$$

$$,\left[\sum\frac{1}{b^2+c^2}\right]\geq\frac{9}{2\sum a^2}(\,Bergstrom)\geq\frac{1}{2R^2}Leibniz$$

$$2\sum\frac{m_a}{h_b+h_c}\overset{m_a\geq h_a}{\geq}\sum\frac{h_a}{h_b+h_c}\overset{Nesbitt}{\geq}\frac{3}{2}\ \ (1),$$

$$\sum\frac{h_a}{m_b+m_c}\overset{AM-GM}{\geq}3\left(\frac{\prod h_a}{\prod(m_b+m_c)}\right)^{\frac{1}{3}}\overset{AM-GM}{\geq}3(27r^3)^{\frac{1}{3}}\frac{1}{\frac{2\sum m_a}{3}}\geq\frac{9r}{3R}=\frac{3r}{R}$$

$$2\sum\frac{m_a^2}{m_b^2+m_c^2}\leq 2\sum\frac{m_a^2}{h_b^2+h_c^2}=\frac{2(4R^2)}{4}\sum\frac{2(b^2+c^2)-a^2}{a^2c^2+a^2b^2}$$
$$=2R^2\left[\sum\frac{2}{a^2}-\sum\frac{1}{b^2+c^2}\right]\leq 2R^2\left(\frac{2}{4r^2}-\frac{1}{2R^2}\right)(Steining),$$

$$\frac{h_ah_bh_c}{w_aw_bw_c}=\prod\cos\left(\frac{A-B}{2}\right)=\frac{s^2+r^2+2Rr}{8R^2}\geq\frac{9r}{4R}-\frac{r^2}{2R^2}(Gerretsen)$$

$$Let\ \frac{R}{r}=x\geq 2\ and\ m=2024\ ,we\ need\ to\ show\ \frac{3}{2}+\frac{3}{x}+\frac{9}{4}x^m-\frac{1}{2}x^{m-1}$$
$$\geq 2^{m+1}+x^2-1\ or\ 9x^{m+1}-x2^{m+3}+10x-2x^m-4x^3+12\geq 0,$$

$$Let\ f(x)=\ 9x^{m+1}-x2^{m+3}+10x-2x^m-4x^3+12,$$
$$f'(x)=9(m+1)x^m-2^{m+3}+10-2mx^{m-1}-12x^2$$
$$=2m(x^m-x^{m-1})+(x^m-12x^2)+(7mx^m+8x^m-2^{m+3})>0\ as\ x\geq 2\ ,$$
$$so\ \ f(x)\ increasing.\,f(2)=0$$

$$so\ f(x)\geq 0\ \ or\ 9x^{m+1}-x2^{m+3}+10x-2x^m-4x^3+12\geq 0,(true)$$