ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{a^{2024}}{b^{2024}+c^{2024}}+\frac{R^{2026}}{r^{2026}}\geq 2^{2026}+\sum_{cyc}\frac{a^{2025}}{b^{2025}+c^{2025}}$$

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$$\sum_{cyc}\frac{a^{2025}}{b^{2025}+c^{2025}}\overset{\text{Reverse Bergstrom}}{\leq}\frac{1}{4}\sum_{cyc}\left(\frac{a^{2025}}{b^{2025}}+\frac{a^{2025}}{c^{2025}}\right)$$

$$\therefore\sum_{cyc}\frac{a^{2025}}{b^{2025}+c^{2025}}\leq\frac{1}{4}\sum_{cyc}\left(\frac{b^{2025}}{c^{2025}}+\frac{c^{2025}}{b^{2025}}\right)\to(1)$$

$$\left(x+\frac{1}{x}\right)^{2025}=\binom{2025}{0}x^{2025}+\binom{2025}{1}x^{2024}.\frac{1}{x}+\binom{2025}{2}x^{2023}.\frac{1}{x^2}+\cdots$$
$$+\binom{2025}{2012}x^{1013}.\frac{1}{x^{1012}}+\binom{2025}{2013}x^{1012}.\frac{1}{x^{1013}}+\cdots$$
$$+\binom{2025}{2023}x^{2}.\frac{1}{x^{2023}}+\binom{2025}{2024}x.\frac{1}{x^{2024}}+\binom{2025}{2025}\frac{1}{x^{2025}}$$
$$=x^{2025}+\frac{1}{x^{2025}}+\binom{2025}{1}.\left(x^{2023}+\frac{1}{x^{2023}}\right)+\binom{2025}{2}.\left(x^{2021}+\frac{1}{x^{2021}}\right)+\cdots$$
$$+\binom{2025}{2012}.\left(x+\frac{1}{x}\right)\left(\because\binom{n}{r}=\binom{n}{n-r}\right)$$
$$\overset{A-G}{\geq}x^{2025}+\frac{1}{x^{2025}}+2\left(\binom{2025}{1}+\binom{2025}{2}+\cdots+\binom{2025}{2012}\right)=x^{2025}+\frac{1}{x^{2025}}+$$
$$\left(\binom{2025}{1}+\binom{2025}{2}+\cdots+\binom{2025}{2012}+\binom{2025}{2013}+\cdots+\binom{2025}{2023}+\binom{2025}{2024}\right)$$
$$\left(\because 2\binom{n}{r}=\binom{n}{r}+\binom{n}{n-r}\right)=x^{2025}+\frac{1}{x^{2025}}$$
$$+\left(\binom{2025}{0}+\binom{2025}{1}+\binom{2025}{2}+\cdots+\binom{2025}{2023}+\binom{2025}{2024}+\binom{2025}{2025}-\left(\binom{2025}{0}+\binom{2025}{2025}\right)\right)$$
$$=x^{2025}+\frac{1}{x^{2025}}+\left(2^{2025}-2\right)\left(\because\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n\right)\text{ and putting }x=\frac{b}{c},$$

we get : $\left(\frac{b}{c}+\frac{c}{b}\right)^{2025}\geq\frac{b^{2025}}{c^{2025}}+\frac{c^{2025}}{b^{2025}}+2^{2025}-2\Rightarrow\frac{b^{2025}}{c^{2025}}+\frac{c^{2025}}{b^{2025}}\leq$

$$\left(\frac{b}{c}+\frac{c}{b}\right)^{2025}-2^{2025}+2\overset{\text{Bandila}}{\leq}\left(\frac{R}{r}\right)^{2025}-2^{2025}+2\text{ and analogs}$$

$$\Rightarrow \frac{1}{4}\sum_{cyc}\left(\frac{b^{2025}}{c^{2025}}+\frac{c^{2025}}{b^{2025}}\right) \leq \frac{3}{4}\left(\frac{R}{r}\right)^{2025} - \frac{3}{4}.2^{2025}+\frac{3}{2} \to (2) \therefore (1),(2) \Rightarrow$$

$$\boxed{\sum_{cyc}\frac{a^{2025}}{b^{2025}+c^{2025}} \leq \frac{3}{4}\left(\frac{R}{r}\right)^{2025} - \frac{3}{4}.2^{2025}+\frac{3}{2} \to (i)} \ \boldsymbol{and} \sum_{cyc}\frac{a^{2024}}{b^{2024}+c^{2024}}+\frac{R^{2026}}{r^{2026}}$$

$$-2^{2026} \overset{Nesbitt}{\geq} \frac{3}{2}+\left(\frac{R}{r}\right)^{2026} - 2^{2026} \overset{Euler}{\geq} \frac{3}{2}+2\left(\left(\frac{R}{r}\right)^{2025} - 2^{2025}\right)$$

$$\geq \frac{3}{2}+\frac{3}{4}\left(\left(\frac{R}{r}\right)^{2025} - 2^{2025}\right)\left(\because \frac{R}{r} \overset{Euler}{\geq} 2 \Rightarrow \left(\frac{R}{r}\right)^{2025} - 2^{2025} \geq 0\right) \overset{via\ (i)}{\geq}$$

$$\sum_{cyc}\frac{a^{2025}}{b^{2025}+c^{2025}} \therefore \sum_{cyc}\frac{a^{2024}}{b^{2024}+c^{2024}}+\frac{R^{2026}}{r^{2026}} \geq 2^{2026}+\sum_{cyc}\frac{a^{2025}}{b^{2025}+c^{2025}}$$

$\forall \Delta ABC$, " = " iff ΔABC is equilateral (QED)