ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationships hold :

$$① \prod_{cyc} \sqrt[2025]{\frac{b+c}{a}} + \left(\frac{R}{2r}\right)^{2025} \ge 1 + \prod_{cyc} \sqrt[2025]{\frac{\cos\frac{A}{2}+\cos\frac{B}{2}}{\cos\frac{C}{2}}} \text{ and}$$

$$② \prod_{cyc} \sqrt[2025]{\frac{\cos\frac{A}{2}+\cos\frac{B}{2}}{\cos\frac{C}{2}}} + \left(\frac{R}{2r}\right)^{2025} \ge 1 + \prod_{cyc} \sqrt[2025]{\frac{b+c}{a}}$$

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$$\prod_{cyc} \frac{\cos\frac{A}{2}+\cos\frac{B}{2}}{\cos\frac{C}{2}} = \frac{\left(\sum_{cyc}\cos\frac{A}{2}\right)\left(\sum_{cyc}\cos\frac{A}{2}\cos\frac{B}{2}\right) - \prod_{cyc}\cos\frac{A}{2}}{\prod_{cyc}\cos\frac{A}{2}}$$

$$\overset{\text{Jensen}}{\le} \frac{\frac{3\sqrt{3}}{2}.\left(\sum_{cyc}\cos^2\frac{A}{2}\right)}{\frac{s}{4R}} - 1 = \frac{\frac{3\sqrt{3}}{2}.\left(\frac{4R+r}{2R}\right)}{\frac{s}{4R}} - 1 \overset{\text{Mitrinovic}}{\le} \frac{3\sqrt{3}(4R+r)}{3\sqrt{3}r} - 1$$

$$\Rightarrow \prod_{cyc} \sqrt[2025]{\frac{\cos\frac{A}{2}+\cos\frac{B}{2}}{\cos\frac{C}{2}}} \le \sqrt[2025]{\frac{4R}{r}} \to (1)$$

$$\text{Again, } \prod_{cyc} \sqrt[2025]{\frac{b+c}{a}} = \prod_{cyc} \sqrt[2025]{\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}}} \overset{0<\cos\frac{B-C}{2}\le 1 \text{ and analogs}}{\le}$$

$$\sqrt[2025]{\frac{1}{\prod_{cyc}\sin\frac{A}{2}}} \Rightarrow \prod_{cyc} \sqrt[2025]{\frac{b+c}{a}} \le \sqrt[2025]{\frac{4R}{r}} \to (2)$$

$$\text{Now, } \sqrt[2025]{8} + \left(\frac{R}{2r}\right)^{2025} \overset{?}{\ge} 1 + \sqrt[2025]{\frac{4R}{r}} \Leftrightarrow \sqrt[2025]{8} + \left(\frac{R}{2r}\right)^{2025} \overset{?}{\ge} 1 + \sqrt[2025]{8}.\sqrt[2025]{\frac{R}{2r}}$$

$$\Leftrightarrow t^{4100625} - 1 \underset{(*)}{\overset{?}{\ge}} \sqrt[2025]{8}.(t-1) \quad \left(t = \sqrt[2025]{\frac{R}{2r}}\right)$$

$$\text{We have : } t^{4100625} - 1 = (t-1)(t^{4100624} + t^{4100623} + \cdots + 1) \overset{\text{Euler}}{\ge}$$

$$(t-1)(4100624) \geq \sqrt[2025]{8}.(t-1)\left(\because t-1=\sqrt[2025]{\frac{R}{2r}}-1 \overset{\text{Euler}}{\geq} 0\right) \Rightarrow (*) \text{ is true}$$

$$\therefore \sqrt[2025]{8}+\left(\frac{R}{2r}\right)^{2025} \geq 1+\sqrt[2025]{\frac{4R}{r}} \rightarrow (3) \text{ and also,}$$

$$\prod_{cyc}\sqrt[2025]{\frac{b+c}{a}}, \prod_{cyc}\sqrt[2025]{\frac{\cos\frac{A}{2}+\cos\frac{B}{2}}{\cos\frac{C}{2}}} \overset{\text{Cesaro}}{\geq} \sqrt[2025]{8} \rightarrow (4) \therefore (1),(2) \text{ and } (4)$$

$\Rightarrow$ **in order to prove ① *and* ②, it suffices to prove :**

$$\sqrt[2025]{8}+\left(\frac{R}{2r}\right)^{2025} \geq 1+\sqrt[2025]{\frac{4R}{r}} \rightarrow \text{true via } (3)$$

$\therefore$ ① *and* ② *are* **both true** $\forall \Delta ABC$, " = " **iff** ΔABC **is equilateral (QED)**