ROMANIAN MATHEMATICAL MAGAZINE

In $any\ \Delta$ ABC, the $following$ $relationship$ holds :

$$\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}+\frac{R^{n+1}}{r^{n+1}}\ge 2^{n+1}+\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}$$

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$$\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}-\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}=\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}.1.1.1}-\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}.1.1}$$

$$\overset{\substack{G-H\\and\\A-G}}{\ge}\sum_{cyc}\frac{4}{\frac{b^3+c^3}{2a^3}+1+1+1}-\sum_{cyc}\frac{\frac{2a^2}{b^2+c^2}+1+1}{3}$$

$$=\sum_{cyc}\frac{8a^3}{b^3+c^3+6a^3}-\frac{2}{3}\left(\sum_{cyc}a^2\right)\sum_{cyc}\frac{1}{b^2+c^2}\overset{\substack{Holder\\and\\A-G\ and\ Leibnitz}}{\ge}$$

$$\frac{8(2s)^3}{3\left(2\sum_{cyc}a^3+6\sum_{cyc}a^3\right)}-\frac{2}{3}.(9R^2).\sum_{cyc}\frac{a}{2abc}$$

$$=\frac{4s^2}{3(s^2-6Rr-3r^2)}-\frac{2}{3}.(9R^2).\frac{2s}{8Rrs}=\frac{4s^2}{3(s^2-6Rr-3r^2)}-\frac{3R}{2r}\overset{?}{\ge}-\frac{R^3-8r^3}{r^3}$$

$$\Leftrightarrow(6R^3-9Rr^2-40r^3)s^2\underbrace{\overset{?}{\ge}}_{(*)}r(36R^4+18R^3r-54R^2r^2-315Rr^3-144r^4)$$

$\boxed{Case\ 1}$ $6R^3-9Rr^2-40r^3\ge 0$ and then : LHS of $(*)\overset{Gerretsen}{\ge}$

$$(6R^3-9Rr^2-40r^3)(16Rr-5r^2)\overset{?}{\ge}r\begin{pmatrix}36R^4+18R^3r-54R^2r^2\\-315Rr^3-144r^4\end{pmatrix}$$

$$\Leftrightarrow 30t^4-24t^3-45t^2-140t+172\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\Big((t-2)(30t^2+96t+219)+352\Big)\to\text{true}\because t\overset{Euler}{\ge}2\Rightarrow(*)\text{ is true}$$

$\boxed{Case\ 2}$ $6R^3-9Rr^2-40r^3<0$ and then : LHS of $(*)=-(40r^3+9Rr^2-6R^3)s^2$

$$\overset{Gerretsen}{\ge}-(40r^3+9Rr^2-6R^3)(4R^2+4Rr+3r^2)$$

$$\overset{?}{\ge}r(36R^4+18R^3r-54R^2r^2-315Rr^3-144r^4)$$

$$\Leftrightarrow 12t^5-6t^4-18t^3-71t^2+64t+12\overset{?}{\ge}0$$

$$\Leftrightarrow(t-2)\Big((t-2)(12t^3+42t^2+102t+169)+332\Big)\to\text{true}\because t\overset{Euler}{\ge}2$$

$$\Rightarrow(*)\text{ is true } and \text{ combining } cases\ 1\ and\ 2,\ (*)\text{ is true }\forall\,\Delta\,ABC$$

$$\therefore\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}-\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\ge-\frac{R^3-8r^3}{r^3}\to(1)$$

Let $f(n)=t^{n+1}-2^{n+1}\ \forall\, t=\frac{R}{r}\ge 2$ and $\forall\, n\ge 2$ and then :

$$f'(n)=t^{n+1}.(\ln t)-2^{n+1}.(\ln 2)\ge 0\because t^{n+1}\ge 2^{n+1}\ and\ \ln t\ge\ln 2$$

$$\Rightarrow t^{n+1}.(\ln t) - 2^{n+1}.(\ln 2) \geq 0 \therefore f(n) \text{ is } \uparrow \forall\, n \geq 2 \Rightarrow f(n) \geq f(2) = \left(\frac{R}{r}\right)^3 - 8$$

$$\Rightarrow \frac{R^3 - 8r^3}{r^3} \leq \left(\frac{R}{r}\right)^{n+1} - 2^{n+1} \therefore -\frac{R^3 - 8r^3}{r^3} \geq -\left(\frac{R^{n+1}}{r^{n+1}} - 2^{n+1}\right) \to (2)$$

$$\therefore (1) \text{ and } (2) \Rightarrow \sum_{cyc} \sqrt[4]{\frac{2a^3}{b^3 + c^3}} - \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2 + c^2}} \geq -\left(\frac{R^{n+1}}{r^{n+1}} - 2^{n+1}\right)$$

$$\therefore \sum_{cyc} \sqrt[4]{\frac{2a^3}{b^3 + c^3}} + \frac{R^{n+1}}{r^{n+1}} \geq 2^{n+1} + \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2 + c^2}}$$

$\forall\, \Delta\, ABC$, " = " iff $\Delta\, ABC$ is equilateral (QED)