ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC and $\forall$ n $\geq$ 2, the following relationship holds :

$$\min\left\{\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}},\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\}+\left(\frac{R}{2r}\right)^{n+1}$$
$$\geq 1+\max\left\{\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}},\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\}$$

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Firstly, we shall prove : $\boxed{\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}+\frac{R^3-8r^3}{8r^3}\overset{(i)}{\geq}\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}}$

$$\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}=\sum_{cyc}\sqrt[4]{\frac{2a}{b+c}\cdot\frac{a^2}{b^2-bc+c^2}\cdot 1\cdot 1}\overset{G-H}{\geq}\sum_{cyc}\frac{4}{\frac{b+c}{2a}+\frac{b^2-bc+c^2}{a^2}+2}$$
$$=\sum_{cyc}\frac{8a^2}{a(b+c)+2b^2-2bc+2c^2+4a^2}\overset{Bergstrom}{\geq}$$
$$\frac{8.4s^2}{2\sum_{cyc}ab+2\sum_{cyc}a^2-2\sum_{cyc}ab+2\sum_{cyc}a^2+4\sum_{cyc}a^2}=\frac{4s^2}{\sum_{cyc}a^2}$$
$$\therefore \sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}\overset{(1)}{\geq}\frac{2s^2}{s^2-4Rr-r^2}$$
$$\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}=\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}\cdot 1\cdot 1}\overset{A-G}{\leq}\sum_{cyc}\frac{\frac{2a^2}{b^2+c^2}+2}{3}=\frac{2}{3}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{1}{b^2+c^2}\right)$$
$$\overset{A-G}{\leq}\frac{2}{3}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{a}{2abc}\right)=\frac{2.2s}{3.8Rrs}\left(\sum_{cyc}a^2\right)\therefore\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}\overset{(2)}{\leq}\frac{s^2-4Rr-r^2}{3Rr}$$

$\therefore$ (1), (2) $\Rightarrow$ in order to prove (i), it suffices to prove :

$$\frac{2s^2}{s^2-4Rr-r^2}+\frac{R^3-8r^3}{8r^3}\overset{(\blacksquare)}{\geq}\frac{s^2-4Rr-r^2}{3Rr}$$
$$\Leftrightarrow\frac{(R^3-8r^3)(s^2-4Rr-r^2)+16r^3s^2}{8r^3(s^2-4Rr-r^2)}\geq\frac{s^2-4Rr-r^2}{3Rr}$$
$$\Leftrightarrow 8r^2s^4-(3R^4+88Rr^3+16r^4)s^2$$
$$+r(12R^5+3R^4r+32R^2r^3+40Rr^4+8r^5)\overset{(*)}{\leq}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ and $s^2-(m+n)\leq 0$, where

$$m=2R^2+10Rr-r^2 \text{ and } n=2(R-2r).\sqrt{R^2-2Rr}$$
$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$
$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\leq 0$$
$$\Rightarrow 8r^2s^4-8r^2s^2(4R^2+20Rr-2r^2)+8r^3(4R+r)^3\leq 0\ \therefore \text{ in order to prove } (*),$$

$$\text{it suffices to prove}: 8r^2s^4-(3R^4+88Rr^3+16r^4)s^2$$
$$+r(12R^5+3R^4r+32R^2r^3+40Rr^4+8r^5)$$
$$\le 8r^2s^4-8r^2s^2(4R^2+20Rr-2r^2)+8r^3(4R+r)^3$$
$$\Leftrightarrow (3R^4-32R^2r^2-72Rr^3+32r^4)s^2 \overset{(**)}{\ge} Rr\begin{pmatrix}12R^4+3R^3r-512R^2r^2\\-352Rr^3-56r^4\end{pmatrix}$$
$$\boxed{Case\ 1}\ 3R^4-32R^2r^2-72Rr^3+32r^4\ge 0 \text{ and then, LHS of } (**)$$
$$\overset{\text{Gerretsen}}{\ge} (3R^4-32R^2r^2-72Rr^3+32r^4)(16Rr-5r^2)$$
$$\overset{?}{\ge} Rr(12R^4+3R^3r-512R^2r^2-352Rr^3-56r^4)$$
$$\Leftrightarrow 18t^5-9t^4-320t^2+464t-80\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left((t-2)(18t^3+63t^2+180t+148)+336\right)\to \text{true} \because t\overset{\text{Euler}}{\ge}2$$
$$\Rightarrow (**) \text{ is true}$$
$$\boxed{Case\ 2}\ 3R^4-32R^2r^2-72Rr^3+32r^4<0 \text{ and then, LHS of } (**)\overset{\text{Rouche}}{\ge}$$
$$-\left(-(3R^4-32R^2r^2-72Rr^3+32r^4)\right)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$
$$\overset{?}{\ge} Rr(12R^4+3R^3r-512R^2r^2-352Rr^3-56r^4)$$
$$\Leftrightarrow 2(R-2r)(3R^5+15R^4r-5R^3r^2+14R^2r^3-108Rr^4+8r^5)\overset{(***)}{\ge}$$
$$2\left(-(3R^4-32R^2r^2-72Rr^3+32r^4)\right)(R-2r).\sqrt{R^2-2Rr}$$
$$\text{Now}, 3R^5+15R^4r-5R^3r^2+14R^2r^3-108Rr^4+8r^5$$
$$=(R-2r)(3R^4+21R^3r+37R^2r^2+88Rr^3+68r^4)+144r^5\overset{\text{Euler}}{\ge}144r^5>0$$
$$\therefore \text{in order to prove } (***), \text{ it suffices to prove}:$$
$$(3R^5+15R^4r-5R^3r^2+14R^2r^3-108Rr^4+8r^5)^2$$
$$>(R^2-2Rr)(3R^4-32R^2r^2-72Rr^3+32r^4)^2\ \left(\because R-2r\overset{\text{Euler}}{\ge}0\right)$$
$$\Leftrightarrow 108t^9+387t^8-18t^7-2283t^6-5508t^5+7596t^4$$
$$+7776t^3+1648t^2+320t+64>0\Leftrightarrow$$
$$(t-2)\left((t-2)\begin{pmatrix}108t^7+819t^6+2826t^5+5745t^4+\\6168t^3+9288t^2+20256t+45520\end{pmatrix}+101376\right)+20736>0$$
$$\to \text{true} \because t\overset{\text{Euler}}{\ge}2\Rightarrow(***)\Rightarrow(**) \text{ is true}$$
$$\therefore \text{combining cases 1 and 2}, (**)\Rightarrow(*)\Rightarrow(\text{i}) \text{ is true } \forall\,\Delta\,ABC$$
$$\text{We shall now prove}: \boxed{\sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}+\frac{R^3-8r^3}{8r^3}\overset{(\text{ii})}{\ge}\sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}}$$
$$\text{We have}: \sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}\overset{A-G}{\le}\sum_{cyc}\sqrt[4]{\frac{2a^3.2}{(b^2+c^2)(b+c)}}=\sum_{cyc}\sqrt[4]{\frac{2a^2}{b^2+c^2}.\frac{2a}{b+c}.1.1}$$
$$\overset{A-G}{\le}\frac{1}{4}\left(\sum_{cyc}\frac{2a^2}{b^2+c^2}+2+2\sum_{cyc}\frac{a}{b+c}\right)\overset{A-G}{\le}\frac{1}{2}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{a}{2abc}\right)+\frac{1}{2}\sum_{cyc}\frac{a}{b+c}$$
$$\therefore \sum_{cyc}\sqrt[4]{\frac{2a^3}{b^3+c^3}}\overset{(3)}{\le}\frac{s^2-4Rr-r^2}{4Rr}+\frac{1}{2}\sum_{cyc}\frac{a}{b+c} \text{ and } \sum_{cyc}\sqrt[3]{\frac{2a^2}{b^2+c^2}}$$

$$= \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2+c^2}.1.1} \overset{G-H}{\geq} \sum_{cyc} \frac{3}{\frac{b^2+c^2}{2a^2}+2} = \sum_{cyc} \frac{6a^2}{b^2+c^2+4a^2} \overset{Bergstrom}{\geq}$$

$$\frac{6.4s^2}{2\sum_{cyc} a^2 + 4\sum_{cyc} a^2} \therefore \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2+c^2}} \overset{(4)}{\geq} \frac{2s^2}{s^2-4Rr-r^2} \therefore (3),(4) \Rightarrow$$

in order to prove (ii), it suffices to prove :

$$\frac{2s^2}{s^2-4Rr-r^2} + \frac{R^3-8r^3}{8r^3} \geq \frac{s^2-4Rr-r^2}{4Rr} + \frac{1}{2}\sum_{cyc} \frac{a}{b+c} \text{ and } \because \text{ via } (\blacksquare),$$

$$\frac{2s^2}{s^2-4Rr-r^2} + \frac{R^3-8r^3}{8r^3} \geq \frac{s^2-4Rr-r^2}{3Rr} \therefore \text{ it suffices to prove :}$$

$$\frac{s^2-4Rr-r^2}{3Rr} \geq \frac{s^2-4Rr-r^2}{4Rr} + \frac{1}{2}\sum_{cyc} \frac{a}{b+c} \Leftrightarrow \frac{s^2-4Rr-r^2}{12Rr} \geq \frac{1}{2}\sum_{cyc} \frac{2s-(b+c)}{b+c}$$

$$\Leftrightarrow \frac{s^2-4Rr-r^2}{6Rr} \geq \frac{2s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)} - 3$$

$$\Leftrightarrow s^4 - 14Rrs^2 + r^2(4R^2+6Rr-r^2) \overset{(\bullet)}{\geq} 0$$

$$\text{Now, LHS of } (\bullet) \overset{Gerretsen}{\geq} (2Rr-5r^2)s^2 + r^2(4R^2+6Rr-r^2)$$

$$= (2Rr-4r^2)s^2 - r^2s^2 + r^2(4R^2+6Rr-r^2) \overset{Gerretsen}{\geq}$$

$$(2Rr-4r^2)(16Rr-5r^2) - r^2(4R^2+4Rr+3r^2) + r^2(4R^2+6Rr-r^2)$$

$$= 8r^2(R-2r)(4R-r) \overset{Euler}{\geq} 0 \Rightarrow (\bullet) \Rightarrow \text{(ii) is true} \therefore \text{(i) and (ii)} \Rightarrow$$

$$\min\left\{\sum_{cyc} \sqrt[4]{\frac{2a^3}{b^3+c^3}}, \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\} + \frac{R^3-8r^3}{8r^3}$$

$$\geq \max\left\{\sum_{cyc} \sqrt[4]{\frac{2a^3}{b^3+c^3}}, \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\}$$

$$\Rightarrow \boxed{\frac{R^3-8r^3}{8r^3} \overset{(iii)}{\geq} \max\left\{\sum_{cyc} \sqrt[4]{\frac{2a^3}{b^3+c^3}}, \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\} - \min\left\{\sum_{cyc} \sqrt[4]{\frac{2a^3}{b^3+c^3}}, \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\}}$$

$$\text{Now, } n \geq 2 \text{ and Euler} \Rightarrow (n+1-3).\ln\frac{R}{2r} \geq 0 \Rightarrow \left(\frac{R}{2r}\right)^{n+1} \geq \left(\frac{R}{2r}\right)^3$$

$$\Rightarrow \left(\frac{R}{2r}\right)^{n+1} - 1 \geq \frac{R^3-8r^3}{8r^3} \overset{\text{via (iii)}}{\geq}$$

$$\max\left\{\sum_{cyc} \sqrt[4]{\frac{2a^3}{b^3+c^3}}, \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\} - \min\left\{\sum_{cyc} \sqrt[4]{\frac{2a^3}{b^3+c^3}}, \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\}$$

$$\therefore \min\left\{\sum_{cyc} \sqrt[4]{\frac{2a^3}{b^3+c^3}}, \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\} + \left(\frac{R}{2r}\right)^{n+1}$$

$$\geq 1 + \max\left\{\sum_{cyc} \sqrt[4]{\frac{2a^3}{b^3+c^3}}, \sum_{cyc} \sqrt[3]{\frac{2a^2}{b^2+c^2}}\right\}$$

$\forall\, \Delta$ ABC and $\forall\, n \geq 2$, " = " iff Δ ABC is equilateral (QED)