Let $n \geq 2$. Then, in Δ ABC, the following relationship holds :

$$\sqrt{\sum_{cyc}\frac{a}{b+c-a}}+\left(\frac{R}{r}\right)^n \geq 2^n+\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c-m_a}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{b+c-a}=\frac{1}{2}\sum_{cyc}\frac{a-s+s}{s-a}=\frac{1}{2}\left(-3+\frac{s(4Rr+r^2)}{r^2s}\right)=\frac{2R-r}{r}$$

$$\Rightarrow \sum_{cyc}\frac{a}{b+c-a}=\frac{2R}{r}-1 \rightarrow (1)$$

Invoking (1) on a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$, whose area via elementary calculations $=\frac{F}{3}$, we arrive at : $\sum_{cyc}\frac{m_a}{m_b+m_c-m_a}$

$$=2.\frac{\frac{2m_a}{3}.\frac{2m_b}{3}.\frac{2m_c}{3}}{\frac{4F}{3}}.\frac{\frac{\frac{2m_a}{3}+\frac{2m_b}{3}+\frac{2m_c}{3}}{2}}{\frac{F}{3}}-1=\frac{4m_am_bm_c(m_a+m_b+m_c)}{9r^2s^2}-1$$

$$\overset{\substack{m_am_bm_c \leq \frac{Rs^2}{2} \\ and \\ m_a+m_b+m_c \leq 4R+r}}{\leq} \frac{2Rs^2(4R+r)}{9r^2s^2}-1 \Rightarrow \sum_{cyc}\frac{m_a}{m_b+m_c-m_a} \leq \frac{8R^2+2Rr-9r^2}{9r^2} \rightarrow (2)$$

(1) and (2) $\Rightarrow$ in order to prove : $\sqrt{\sum_{cyc}\frac{a}{b+c-a}}+\frac{R^2-4r^2}{4r^2} \geq$

$\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c-m_a}}$, it suffices to prove :

$$\sqrt{\frac{2R-r}{r}}+\frac{R^2-4r^2}{4r^2} \geq \sqrt{\frac{8R^2+2Rr-9r^2}{9r^2}} \Leftrightarrow \frac{R^2-4r^2}{4r^2} \geq \frac{\frac{8R^2+2Rr-9r^2}{9r^2}-\frac{2R-r}{r}}{\sqrt{\frac{8R^2+2Rr-9r^2}{9r^2}}+\sqrt{\frac{2R-r}{r}}}$$

$$\Leftrightarrow \frac{(R-2r)(R+2r)}{4r^2} \overset{(*)}{\geq} \frac{3r.8R(R-2r)}{9r^2\left(\sqrt{8R^2+2Rr-9r^2}+\sqrt{9r(2R-r)}\right)}$$

$$8R^2+2Rr-9r^2-27r^2=2(R-2r)(4R+9r) \overset{Euler}{\geq} 0$$

$$\Rightarrow \sqrt{8R^2+2Rr-9r^2} \overset{(i)}{\geq} \sqrt{27r^2}$$

$$and\ 9r(2R-r) \overset{Euler}{\geq} 9r(4r-r) \Rightarrow \sqrt{9r(2R-r)} \overset{(ii)}{\geq} \sqrt{27r^2}$$

Now, $\because R-2r \overset{Euler}{\geq} 0 \therefore$ in order to prove $(*)$, it suffices to prove :

$$\frac{R+2r}{4} > \frac{8Rr}{3\left(\sqrt{8R^2+2Rr-9r^2}+\sqrt{9r(2R-r)}\right)}$$

$$\Leftrightarrow 9(R+2r)^2\left(\sqrt{8R^2+2Rr-9r^2}+\sqrt{9r(2R-r)}\right)^2 > 1024R^2r^2$$

$$\Leftrightarrow 9(R+2r)^2\begin{pmatrix} 8R^2+2Rr-9r^2+9r(2R-r) \\ +2.\sqrt{8R^2+2Rr-9r^2}.\sqrt{9r(2R-r)} \end{pmatrix} \overset{(**)}{>} 1024R^2r^2$$

Again, via (i) and (ii), LHS of $(**) \geq$

$$9(R+2r)^2\left(8R^2+2Rr-9r^2+9r(2R-r)+54r^2\right) \overset{?}{>} 1024R^2r^2$$

$$\Leftrightarrow 72R^4+468R^3r+308R^2r^2+2016Rr^3+1296r^4 \overset{?}{>} 0 \rightarrow \text{true}$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \Rightarrow \sqrt{\sum_{cyc}\frac{a}{b+c-a}}+\frac{R^2-4r^2}{4r^2} \geq \sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c-m_a}}$$

$$\Rightarrow \frac{R^2-4r^2}{4r^2} \geq \sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c-m_a}}-\sqrt{\sum_{cyc}\frac{a}{b+c-a}} \rightarrow (3)$$

Let $f(n)=t^n-2^n\ \forall\ t=\frac{R}{r}\geq 2$ ($t\rightarrow$ fixed) and $\forall\ n\geq 2$ and then :

$f'(n)=t^n.(\ln t)-2^n.(\ln 2)\geq 0 \because t^n\geq 2^n$ and $\ln t\geq \ln 2 \Rightarrow t^n.(\ln t)-2^n.(\ln 2)$

$$\geq 0 \therefore f(n) \text{ is } \uparrow \forall\ n\geq 2 \Rightarrow f(n)\geq f(2)=\left(\frac{R}{r}\right)^2-4 \Rightarrow \left(\frac{R}{r}\right)^n-2^n \geq \frac{R^2-4r^2}{4r^2}$$

$$\overset{\text{via }(3)}{\geq} \sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c-m_a}}-\sqrt{\sum_{cyc}\frac{a}{b+c-a}}$$

$$\therefore \sqrt{\sum_{cyc}\frac{a}{b+c-a}}+\left(\frac{R}{r}\right)^n \geq 2^n+\sqrt{\sum_{cyc}\frac{m_a}{m_b+m_c-m_a}}$$

$\forall\ \Delta$ ABC and $\forall\ n\geq 2$, " = " iff Δ ABC is equilateral (QED)

Proof of $m_a m_b m_c \leq \frac{Rs^2}{2}$

$$m_a^2m_b^2m_c^2=\frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$\overset{(1)}{=}\frac{1}{64}\left\{-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right\}$$

$$\text{Now}, \sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3-3\left(2a^2b^2c^2+\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore \sum_{cyc} a^6 \overset{(2)}{=} \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right) \overset{(3)}{=}$$

$$\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \therefore (1),(2),(3) \Rightarrow m_a^2m_b^2m_c^2$$

$$= \frac{1}{64}\begin{pmatrix} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{pmatrix}$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left\{\begin{matrix} -32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2 \\ -576Rrs^2(s^2-4Rr-r^2) - 432R^2r^2s^2 \end{matrix}\right\}$$

$$= \frac{1}{16}\{s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3\}$$

$$\le \frac{R^2s^4}{4} \Leftrightarrow$$

$$s^6 - s^4(4R^2+12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3 \overset{(\bullet)}{\le} 0$$

$$\text{Now, LHS of } (\bullet) \overset{\text{Gerretsen}}{\le} -s^4(8Rr-36r^2) - s^2(60R^2r^2+120Rr^3+33r^4)$$

$$-r^3(4R+r)^3 \overset{?}{\le} 0$$

$$\Leftrightarrow s^4(8R-16r) + s^2(60R^2r+120Rr^2+33r^3) + r^2(4R+r)^3 \underset{(\bullet\bullet)}{\overset{?}{\ge}} 20rs^4$$

$$\text{Now, LHS of } (\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\ge}}_{(a)} s^2(16Rr-5r^2)(8R-16r)$$

$$+s^2(60R^2r+120Rr^2+33r^3) + r^2(4R+r)^3 \textit{ and}$$

$$\text{RHS of } (\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\le}}_{(b)} 20rs^2(4R^2+4Rr+3r^2)$$

$(a),(b) \Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :

$$s^2(16Rr-5r^2)(8R-16r) + s^2(60R^2r+120Rr^2+33r^3) + r^2(4R+r)^3$$

$$\ge 20rs^2(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow s^2(108R^2-256Rr+53r^2) + r(4R+r)^3 \ge 0$$

$$\Leftrightarrow s^2\left(108R^2-256Rr+80r^2\right)+r(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\geq} 27r^2s^2$$

$$\text{Now, LHS of }(\bullet\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\geq}}_{(c)}\left(108R^2-256Rr+80r^2\right)\left(16Rr-5r^2\right)+r(4R+r)^3$$

$$\text{and RHS of }(\bullet\bullet\bullet)\underbrace{\overset{\text{Geretsen}}{\leq}}_{(d)}27r^2\left(4R^2+4Rr+3r^2\right)$$

$$(c),(d)\Rightarrow\text{ in order to prove }(\bullet\bullet\bullet),\text{ it suffices to prove :}$$

$$\left(108R^2-256Rr+80r^2\right)\left(16Rr-5r^2\right)+r(4R+r)^3\geq 27r^2\left(4R^2+4Rr+3r^2\right)$$

$$\Leftrightarrow 224t^3-587t^2+308t-60\geq 0\ \left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(224t+309)+648\right)\geq 0\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)$$

$$\Rightarrow(\bullet)\text{ is true}\Rightarrow m_a^2m_b^2m_c^2\leq\frac{R^2s^4}{4}\Rightarrow m_am_bm_c\leq\frac{Rs^2}{2}\ (\text{QED})$$