ROMANIAN MATHEMATICAL MAGAZINE

Let $n \geq 2$. In ΔABC the following relationship holds

$$\sum_{cyc}\sqrt[3]{\frac{a+b}{a+b-c}}+\left(\frac{R}{r}\right)^n \geq 2^n+\sum_{cyc}\sqrt[3]{\frac{m_a+m_b}{m_a+m_b-m_c}}.$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\sum_{cyc}\sqrt[3]{\frac{a+b}{a+b-c}} \geq 3\sqrt[9]{\frac{(a+b)(b+c)(c+a)}{(a+b-c)(b+c-a)(c+a-b)}} \overset{\substack{Cesaro\\Padoa}}{\geq} 3\sqrt[9]{\frac{8abc}{abc}}=3\sqrt[3]{2}.$$

Since $(m_a+m_b+m_c)(m_a+m_b-m_c)(m_a-m_b+m_c)(-m_a+m_b+m_c)=9F^2$, then

$$\sum_{cyc}\sqrt[3]{\frac{m_a+m_b}{m_a+m_b-m_c}}=\sum_{cyc}\sqrt[3]{\frac{(m_a+m_b+m_c)(m_a+m_b)(m_b+m_c-m_a)(m_c+m_a-m_b)}{9F^2}}$$

$$\overset{AM-GM}{\leq}\sqrt[3]{\frac{m_a+m_b+m_c}{9F^2.4}}.\sum_{cyc}\frac{(m_a+m_b)+2(m_b+m_c-m_a)+2(m_c+m_a-m_b)}{3}$$

$$=\sqrt[3]{\frac{2(m_a+m_b+m_c)^4}{9s^2r^2}} \overset{\substack{Gotman\\Mitrinovic}}{\leq} \sqrt[3]{\frac{2\left(\frac{9R}{2}\right)^4}{9.27r^2.r^2}}=\frac{3}{2}\sqrt[3]{\left(\frac{R}{r}\right)^4}.$$

So it suffices to prove that

$$3\sqrt[3]{2}+\left(\frac{R}{r}\right)^n \geq 2^n+\frac{3}{2}\sqrt[3]{\left(\frac{R}{r}\right)^4} \quad \text{or} \quad \sqrt[3]{\left(\frac{R}{r}\right)^4}.\left(\left(\frac{R}{r}\right)^{n-\frac{4}{3}}-\frac{3}{2}\right) \geq 2^n-3\sqrt[3]{2},$$

which is true because, $\sqrt[3]{\left(\frac{R}{r}\right)^4} \geq \sqrt[3]{2^4}=2\sqrt[3]{2}$ and $\left(\frac{R}{r}\right)^{n-\frac{4}{3}}-\frac{3}{2} \geq 2^{n-\frac{4}{3}}-\frac{3}{2} \overset{n\geq 2}{>} 0.$

So the proof is complete. Equality holds iff ΔABC is equilateral.