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Let $n\ge 2$. Then, in ΔABC, the following relationship holds :

$$2023\sum_{cyc}\sqrt[2023]{\frac{n_a}{n_b+n_c}}+\left(\frac{R}{r}\right)^n\ge 2^n+2023\sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2$$
$$=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\therefore n_a^2=s^2-2r_ah_a$$
$$\Rightarrow a^2n_a^2\overset{?}{\le}4(R-r)^2s^2a^2(s^2-2h_ar_a)\overset{?}{\le}4(R-r)^2s^2$$
$$\Leftrightarrow(4R^2\sin^2A)s^2-4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right)\overset{?}{\le}4(R^2-2Rr+r^2)s^2$$
$$\Leftrightarrow R^2(1-\sin^2A)-2Rr\left(1-2\sin^2\frac{A}{2}\right)+r^2\overset{?}{\ge}0\Leftrightarrow R^2\cos^2A-2Rr\cos A+r^2\overset{?}{\ge}0$$
$$\Leftrightarrow(R\cos A-r)^2\overset{?}{\ge}0\to\text{true}\therefore an_a\le 2Rs-2rs\Rightarrow\frac{n_a}{h_a}\le\frac{2Rs}{a\left(\frac{2rs}{a}\right)}-\frac{2rs}{a\left(\frac{2rs}{a}\right)}$$
$$\Rightarrow\frac{n_a}{h_a}\le\frac{R}{r}-1\text{ and analogs}\therefore 2023\sum_{cyc}\sqrt[2023]{\frac{n_a}{n_b+n_c}}+\frac{R^2-4r^2}{r^2}$$
$$\ge 2023\sum_{cyc}\sqrt[2023]{\frac{m_a}{\left(\frac{R}{r}-1\right)(h_b+h_c)}}+\frac{R^2-4r^2}{r^2}$$
$$\ge 2023\sum_{cyc}\sqrt[2023]{\frac{m_a}{\left(\frac{R}{r}-1\right)(m_b+m_c)}}+\frac{R^2-4r^2}{r^2}\overset{?}{\ge}2023\sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}$$
$$\Leftrightarrow\boxed{\frac{R^2-4r^2}{r^2}\underset{(*)}{\overset{?}{\ge}}\frac{2023}{\sqrt[2023]{2}}\left(1-\frac{1}{\sqrt[2023]{\frac{R}{r}-1}}\right)\sum_{cyc}\sqrt[2023]{\frac{2m_a}{m_b+m_c}}}$$

$$\text{Now,}\sum_{cyc}\sqrt[2023]{\frac{2a}{b+c}}=\sum_{cyc}\sqrt[2023]{\frac{2a}{b+c}\cdot\underbrace{1.1\dots1}_{2022\text{ terms}}}\overset{A-G}{\le}\frac{1}{2023}\sum_{cyc}\left(\frac{2a}{b+c}+2+2020\right)$$

$$=\frac{1}{2023}\left(2\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{1}{b+c}\right)+6060\right)$$

$$=\frac{1}{2023}\left(2(2s)\frac{5s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}+6060\right)$$

$$=\frac{1}{2023}\left(\frac{10(s^2+2Rr+r^2)}{s^2+2Rr+r^2}-\frac{12Rr+8r^2}{s^2+2Rr+r^2}+6060\right)\overset{\text{Gerretsen}}{\leq}$$

$$\frac{1}{2023}\left(9+1-\frac{12Rr+8r^2}{4R^2+4Rr+3r^2+2Rr+r^2}+6060\right)$$

$$=\frac{1}{2023}\left(9+\frac{2R^2-3Rr-2r^2}{2R^2+3Rr+2r^2}+6060\right)\leq\frac{1}{2023}\left(9+\frac{R-2r}{r}+6060\right)$$

$$\left(\begin{array}{c}\frac{2R^2-3Rr-2r^2}{2R^2+3Rr+2r^2}\overset{?}{\leq}\frac{R-2r}{r}\Leftrightarrow 2\sigma^3-3\sigma^2-\sigma-2\overset{?}{\geq}0\ \left(\sigma=\frac{R}{r}\right)\\ \Leftrightarrow(\sigma-2)(2\sigma^2+\sigma+1)\overset{?}{\geq}0\to\text{true}\because\sigma\overset{\text{Euler}}{\geq}2\end{array}\right)$$

$$\boxed{\sum_{cyc}\sqrt[2023]{\frac{2a}{b+c}}\leq\frac{1}{2023}\left(\frac{R}{r}+6067\right)}$$ *and invoking it on a triangle with sides* $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$,

whose *area* via elementary *calculations* $=\frac{F}{3}$, we *arrive at* : $\sum_{cyc}\sqrt[2023]{\frac{2m_a}{m_b+m_c}}$

$$\leq\frac{1}{2023}\left(\frac{\frac{2m_a}{3}\cdot\frac{2m_b}{3}\cdot\frac{2m_c}{3}}{\frac{4F}{3}}\cdot\frac{\frac{\frac{2m_a}{3}+\frac{2m_b}{3}+\frac{2m_c}{3}}{2}}{\frac{F}{3}}+6067\right)$$

$$=\frac{1}{2023}\left(\frac{2m_am_bm_c(m_a+m_b+m_c)}{9r^2s^2}+6067\right)\overset{\substack{m_am_bm_c\leq\frac{Rs^2}{2}\\ \text{and}\\ m_a+m_b+m_c\leq 4R+r}}{\leq}$$

$$\frac{1}{2023}\left(\frac{Rs^2(4R+r)}{9r^2s^2}+6067\right)\Rightarrow\boxed{\sum_{cyc}\sqrt[2023]{\frac{2m_a}{m_b+m_c}}\leq\frac{1}{2023}\left(\frac{4\sigma^2}{9}+\frac{\sigma}{9}+6067\right)}$$

$$\Rightarrow\text{RHS of }(*)\leq\frac{1}{\sqrt[2023]{2}}\left(1-\frac{1}{t}\right)\left(\frac{4\sigma^2}{9}+\frac{\sigma}{9}+6067\right)\left(t=\sqrt[2023]{\frac{R}{r}-1}\overset{\text{Euler}}{\geq}1\right)$$

$$\leq\left(\frac{t-1}{t}\right)\left(\frac{4\sigma^2}{9}+\frac{\sigma}{9}+6067\right)\overset{?}{\leq}\frac{R^2-4r^2}{r^2}=\sigma^2-4$$

$$\Leftrightarrow(\sigma^2-4)t\overset{?}{\geq}(t-1)\left(\frac{4\sigma^2}{9}+\frac{\sigma}{9}+6067\right)$$

$$\Leftrightarrow t\left(\frac{5\sigma^2}{9}-\frac{\sigma}{9}-2-6069\right)+\frac{4\sigma^2}{9}+\frac{\sigma}{9}-2+6069\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{t(5\sigma+9)(\sigma-2)}{9}+\frac{(4\sigma+9)(\sigma-2)}{9}-6069(t-1)\overset{?}{\geq}0$$

$$\Leftrightarrow \frac{t(5\sigma+9)+4\sigma+9}{9}.\left(t^{2023}-1\right)-6069(t-1)\overset{?}{\geq}0$$

$$\left(\because t^{2023}=\frac{R}{r}-1=\sigma-1\Rightarrow\sigma-2=t^{2023}-1\right)$$

$$\Leftrightarrow \boxed{(t-1)\left(\frac{t(5\sigma+9)+4\sigma+9}{9}.\left(t^{2022}+t^{2021}+\cdots+t+1\right)-6069\right)\underbrace{\overset{?}{\geq}}_{(**)}0}$$

$$\text{Now,}\because t\geq 1,\therefore \frac{t(5\sigma+9)+4\sigma+9}{9}.\left(t^{2022}+t^{2021}+\cdots+t+1\right)-6069$$

$$\geq \frac{5\sigma+9+4\sigma+9}{9}.\left(1^{2022}+1^{2021}+\cdots+1+1\right)-6069=2023(\sigma+2-3)$$

$$\Rightarrow \text{LHS of }(**)\geq 2023(t-1)(\sigma-1)\geq 0\because t\geq 1\ \text{and}\ \sigma-1\overset{\text{Euler}}{\geq}1>0\Rightarrow(**)$$

$$\Rightarrow(*)\text{ is true}\therefore\frac{R^2-4r^2}{r^2}\geq 2023\sum_{\text{cyc}}\sqrt[2023]{\frac{m_a}{m_b+m_c}}-2023\sum_{\text{cyc}}\sqrt[2023]{\frac{n_a}{n_b+n_c}}\rightarrow(1)$$

$$\text{Let } f(n)=t^n-2^n\ \forall\ t=\frac{R}{r}\geq 2\ (t\rightarrow\text{fixed})\ \text{and}\ \forall\ n\geq 2\ \text{and then}:$$

$$f'(n)=t^n.(\ln t)-2^n.(\ln 2)\geq 0\because t^n\geq 2^n\ \text{and}\ \ln t\geq\ln 2$$

$$\Rightarrow t^n.(\ln t)-2^n.(\ln 2)\geq 0\therefore f(n)\text{ is}\uparrow\forall\ n\geq 2\Rightarrow f(n)\geq f(2)$$

$$=\left(\frac{R}{r}\right)^2-4\Rightarrow\left(\frac{R}{r}\right)^n-2^n\geq\frac{R^2-4r^2}{r^2}\overset{\text{via }(1)}{\geq}$$

$$2023\sum_{\text{cyc}}\sqrt[2023]{\frac{m_a}{m_b+m_c}}-2023\sum_{\text{cyc}}\sqrt[2023]{\frac{n_a}{n_b+n_c}}$$

$$\therefore 2023\sum_{\text{cyc}}\sqrt[2023]{\frac{n_a}{n_b+n_c}}+\left(\frac{R}{r}\right)^n\geq 2^n+2023\sum_{\text{cyc}}\sqrt[2023]{\frac{m_a}{m_b+m_c}}$$

$\forall\ \Delta\ ABC$ *and* $\forall\ n\geq 2$," = " iff $\Delta\ ABC$ is equi*lateral* (QED)

$$\boxed{\text{Proof of } m_am_bm_c\leq\frac{Rs^2}{2}}$$

$$m_a^2m_b^2m_c^2=\frac{1}{64}\left(2b^2+2c^2-a^2\right)\left(2c^2+2a^2-b^2\right)\left(2a^2+2b^2-c^2\right)$$

$$\overset{(1)}{=}\frac{1}{64}\left\{-4\sum_{\text{cyc}}a^6+6\left(\sum_{\text{cyc}}a^4b^2+\sum_{\text{cyc}}a^2b^4\right)+3a^2b^2c^2\right\}$$

$$\text{Now,}\sum_{\text{cyc}}a^6=\left(\sum_{\text{cyc}}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{\text{cyc}}a^2\right)^3-3\left(2a^2b^2c^2+\sum_{\text{cyc}}\left(a^2b^2\left(\sum_{\text{cyc}}a^2-c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore \sum_{cyc}a^6\overset{(2)}{=}\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4=\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\overset{(3)}{=}$$

$$\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-3a^2b^2c^2\ \therefore (1),(2),(3)\Rightarrow m_a^2m_b^2m_c^2$$

$$=\frac{1}{64}\left(\begin{array}{c}-4\left(\sum_{cyc}a^2\right)^3-12a^2b^2c^2+12\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\\+6\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-18a^2b^2c^2+3a^2b^2c^2\end{array}\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\left(\sum_{cyc}ab\right)^2-16Rrs^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left\{\begin{array}{c}-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2\\-576Rrs^2(s^2-4Rr-r^2)-432R^2r^2s^2\end{array}\right\}$$

$$=\frac{1}{16}\{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\}$$

$$\leq\frac{R^2s^4}{4}\Leftrightarrow$$

$$s^6-s^4(4R^2+12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\overset{(\bullet)}{\leq}0$$

$$\text{Now, LHS of }(\bullet)\overset{\text{Gerretsen}}{\leq}-s^4(8Rr-36r^2)-s^2(60R^2r^2+120Rr^3+33r^4)$$

$$-r^3(4R+r)^3\overset{?}{\leq}0$$

$$\Leftrightarrow s^4(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3\underset{(\bullet\bullet)}{\overset{?}{\geq}}20rs^4$$

$$\text{Now, LHS of }(\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\geq}}_{(a)}s^2(16Rr-5r^2)(8R-16r)$$

$$+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3\ \text{and}$$

$$\text{RHS of } (\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\leq}}_{(b)} 20rs^2(4R^2+4Rr+3r^2)$$

$(a),(b) \Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :

$$s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3 \geq 20rs^2(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3 \geq 0$$

$$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\geq} 27r^2s^2$$

$$\text{Now, LHS of } (\bullet\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(c)} (108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$$

$$\text{and RHS of } (\bullet\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\leq}}_{(d)} 27r^2(4R^2+4Rr+3r^2)$$

$(c),(d) \Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3 \geq 27r^2(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow 224t^3-587t^2+308t-60 \geq 0 \left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\big((t-2)(224t+309)+648\big) \geq 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet)$$

$$\Rightarrow (\bullet) \text{ is true} \Rightarrow m_a^2m_b^2m_c^2 \leq \frac{R^2s^4}{4} \Rightarrow m_am_bm_c \leq \frac{Rs^2}{2} \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have

$$\sum_{cyc} \sqrt[2023]{\frac{m_a}{m_a+m_b}} \leq \sqrt[2023]{\sum_{cyc} m_a(m_b+m_c) \cdot \sum_{cyc} \frac{1}{(m_a+m_b)(m_b+m_c)} \cdot \left(\sum_{cyc} 1\right)^{2021}}$$

$$= \sqrt[2023]{4.3^{2021} \cdot \frac{(m_a+m_b+m_c)(m_am_b+m_bm_c+m_cm_a)}{(m_a+m_b)(m_b+m_c)(m_c+m_a)}} \overset{AM-GM}{\leq} \sqrt[2023]{4.3^{2021} \cdot \frac{9}{8}}$$

$$= \frac{3}{\sqrt[2023]{2}}.$$

Now, using AM − GM inequality and the known formula $n_a{}^2 = s^2 - 2r_ah_a$, we have

$$n_a \leq \frac{n_a{}^2+r_a{}^2}{2r_a} = \frac{s^2+\left(s\tan\frac{A}{2}\right)^2}{2s\tan\frac{A}{2}} - h_a = \frac{s\sec^2\frac{A}{2}}{2\tan\frac{A}{2}} - h_a = \frac{s}{\sin A} - h_a = \frac{bc}{2r} - \frac{bc}{2R}$$

$$= \frac{(R-r)bc}{2Rr}.$$

Using this result and the AM − GM inequality, we obtain

$$\sum_{cyc}\sqrt[2023]{\frac{n_a}{n_b+n_c}}\ge 3\sqrt[6069]{\frac{n_an_bn_c}{(n_a+n_b)(n_b+n_c)(n_c+n_a)}}$$

$$\ge 3\sqrt[6069]{\frac{(2Rr)^3.m_am_bm_c}{(R-r)^3abc(a+b)(b+c)(c+a)}}$$

$$\overset{m_a\ge\frac{(b+c)}{2}\cos\frac{A}{2}}{\ge} 3\sqrt[6069]{\frac{(Rr)^3.\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{(R-r)^3.abc}}=3\sqrt[6069]{\frac{(Rr)^3.\frac{s}{4R}}{(R-r)^3.4Rsr}}=3\sqrt[6069]{\frac{Rr^2}{16(R-r)^3}}$$

$$\ge 3\sqrt[6069]{\frac{r^3}{8(R-r)^3}}=3\sqrt[2023]{\frac{r}{2(R-r)}}\overset{AM-GM}{\ge}\frac{3}{2023\sqrt[2023]{2}}\left(2024-\frac{R-r}{r}\right).$$

Using these results, it suffices to prove that

$$\frac{3}{\sqrt[2023]{2}}\left(2024-\frac{R-r}{r}\right)+\left(\frac{R}{r}\right)^n\ge 2^n+\frac{2023.3}{\sqrt[2023]{2}}$$

$$\Leftrightarrow\left(\frac{R}{r}-2\right)\left(\frac{R}{r}+2-\frac{3}{\sqrt[2023]{2}}\right)+\left(\frac{R}{r}\right)^2\left(\left(\frac{R}{r}\right)^{n-2}-1\right)-4(2^{n-2}-1)\ge 0.$$

which is true by Euler's inequality,

$R\ge 2r$. Equality holds iff ΔABC is equilateral.