ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationships hold :

$$(a)\ \frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\ge a^2+b^2+c^2+\frac{\sum_{cyc}(b-c)^4}{2023(a^2+b^2+c^2)+2024(a+b+c)^2}\ \text{and}$$

$$(b)\ \sum_{cyc}\sqrt{\frac{m_a}{n_a+g_a-m_a}}+\frac{R^2}{r^2}\ge 4+\sum_{cyc}\sqrt{\frac{a}{b+c-a}}$$

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$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}-(a^2+b^2+c^2)-\frac{\sum_{cyc}(b-c)^4}{2023(a^2+b^2+c^2)+2024(a+b+c)^2}$$

$$=\frac{a^4}{ab}+\frac{b^4}{bc}+\frac{c^4}{ca}-(a^2+b^2+c^2)-\frac{\sum_{cyc}(b-c)^4}{2023(a^2+b^2+c^2)+2024(a+b+c)^2}$$

$$\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}ab}-\sum_{cyc}a^2$$

$$-\frac{2}{\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2}\cdot\left(\sum_{cyc}a^4+3\sum_{cyc}a^2b^2-2\sum_{cyc}\left(ab\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\frac{\sum_{cyc}a^2}{\sum_{cyc}ab}(s^2-12Rr-3r^2)$$

$$-\frac{2}{\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2}\cdot\left(\begin{matrix}\left(\sum_{cyc}a^2\right)^2+\left(\sum_{cyc}ab\right)^2-2abc\sum_{cyc}a\\-2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+2abc\sum_{cyc}a\end{matrix}\right)$$

$$=\frac{\sum_{cyc}a^2}{\sum_{cyc}ab}(s^2-12Rr-3r^2)-\frac{2}{\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2}\cdot\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)^2$$

$$=\frac{2(s^2-4Rr-r^2)}{\sum_{cyc}ab}(s^2-12Rr-3r^2)-\frac{2(s^2-12Rr-3r^2)^2}{\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2}$$

$$=2(s^2-12Rr-3r^2)\left(\frac{s^2-4Rr-r^2}{\sum_{cyc}ab}-\frac{s^2-12Rr-3r^2}{\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2}\right)$$

$$\ge 2(s^2-12Rr-3r^2)\left(\frac{s^2-12Rr-3r^2}{\sum_{cyc}ab}-\frac{s^2-12Rr-3r^2}{\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2}\right)$$

$$\left(\begin{array}{c}\because s^2-4Rr-r^2>s^2-12Rr-3r^2\overset{\text{Euler}}{\geq}16Rr-5r^2-12Rr-3r^2\\=4r(R-2r)\overset{\text{Euler}}{\geq}0\end{array}\right)$$

$$=2\left(s^2-12Rr-3r^2\right)^2\left(\frac{\sum_{cyc}a^2-\sum_{cyc}ab+\left(\sum_{cyc}a\right)^2}{\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2\right)}\right)\geq 0$$

$$\therefore\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\geq a^2+b^2+c^2$$

$$+\frac{\sum_{cyc}(b-c)^4}{2023(a^2+b^2+c^2)+2024(a+b+c)^2},''='' \text{ iff } \Delta\, ABC \text{ is equilateral}$$

$$\text{Now, Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s\left(b^2+c^2\right)-bc(2s-a)=an_a^2+a\left(s^2-s(2s-a)+bc\right)\Rightarrow s\left(b^2+c^2\right)-2sbc$$

$$=an_a^2+a\left(as-s^2\right)\Rightarrow s\left(b^2+c^2-a^2-2bc\right)=an_a^2-as^2\Rightarrow an_a^2=$$

$$as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\therefore n_a^2=s^2-2r_ah_a$$

$$\Rightarrow a^2n_a^2\overset{?}{\leq}4(R-r)^2s^2a^2\left(s^2-2h_ar_a\right)\overset{?}{\leq}4(R-r)^2s^2$$

$$\Leftrightarrow\left(4R^2\sin^2A\right)s^2-4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right)\overset{?}{\leq}4\left(R^2-2Rr+r^2\right)s^2$$

$$\Leftrightarrow R^2\left(1-\sin^2A\right)-2Rr\left(1-2\sin^2\frac{A}{2}\right)+r^2\overset{?}{\geq}0\Leftrightarrow R^2\cos^2A-2Rr\cos A+r^2\overset{?}{\geq}0$$

$$\Leftrightarrow(R\cos A-r)^2\overset{?}{\geq}0\rightarrow\text{true}\therefore an_a\leq 2Rs-2rs\Rightarrow\frac{n_a}{h_a}\leq\frac{2Rs}{a\left(\frac{2rs}{a}\right)}-\frac{2rs}{a\left(\frac{2rs}{a}\right)}$$

$$\Rightarrow\frac{n_a}{h_a}\leq\frac{R}{r}-1\text{ and analogs}\rightarrow(1)$$

$$\text{Triangle inequality}\Rightarrow g_a\leq AI+r\overset{?}{\leq}w_a\Leftrightarrow\frac{r}{\sin\frac{A}{2}}+r\overset{?}{\leq}\frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow\frac{r}{\sin\frac{A}{2}}+r\overset{?}{\leq}\frac{8Rrs\cos\frac{A}{2}}{4R(b+c)\sin\frac{A}{2}\cos\frac{A}{2}}\Leftrightarrow\frac{1}{\sin\frac{A}{2}}+1\overset{?}{\leq}\frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow\frac{1}{\sin\frac{A}{2}}+1\overset{?}{\leq}+\frac{a}{(b+c)\sin\frac{A}{2}}+\frac{1}{\sin\frac{A}{2}}\Leftrightarrow(b+c)\sin\frac{A}{2}\overset{?}{\leq}a$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2}\overset{?}{\leq}4R\sin\frac{A}{2}\cos\frac{A}{2}\Leftrightarrow\cos\frac{B-C}{2}\overset{?}{\leq}1\rightarrow\text{true}$$

$$\therefore g_a\leq w_a\overset{A-G}{\leq}\sqrt{s(s-a)}\leq m_a\text{ and analogs}$$

$$\therefore\sum_{cyc}\sqrt{\frac{m_a}{n_a+g_a-m_a}}\geq\sum_{cyc}\sqrt{\frac{h_a}{n_a}}\overset{\text{via (1)}}{\geq}\sum_{cyc}\sqrt{\frac{h_a}{h_a\left(\frac{R}{r}-1\right)}}$$

$$\Rightarrow\sum_{cyc}\sqrt{\frac{m_a}{n_a+g_a-m_a}}\geq 3\sqrt{\frac{r}{R-r}}\rightarrow(2)$$

$$\text{Also, } \sum_{cyc}\sqrt{\frac{a}{b+c-a}} \overset{CBS}{\leq} \sqrt{\sum_{cyc} a}\cdot\sqrt{\sum_{cyc}\frac{1}{2(s-a)}} = \sqrt{\frac{2s(4Rr+r^2)}{2r^2s}}$$

$$= \sqrt{\frac{4R+r}{r}} \overset{?}{\leq} \frac{R^2-4r^2}{r^2} + 3\sqrt{\frac{r}{R-r}} \Leftrightarrow \frac{R^2-4r^2}{r^2} \overset{?}{\geq} \frac{\frac{4R+r}{r} - \frac{9r}{R-r}}{\sqrt{\frac{4R+r}{r}} + 3\sqrt{\frac{r}{R-r}}}$$

$$\Leftrightarrow \frac{(R-2r)(R+2r)}{r^2} \overset{?}{\geq} \frac{\frac{(4R+5r)(R-2r)}{r(R-r)}}{\sqrt{\frac{4R+r}{r}} + 3\sqrt{\frac{r}{R-r}}} \text{ and } \because R-2r \overset{Euler}{\geq} 0 \therefore \text{it suffices to prove :}$$

$$\frac{(R+2r)(R-r)}{r}\left(\sqrt{\frac{4R+r}{r}} + 3\sqrt{\frac{r}{R-r}}\right) \overset{?}{>} 4R+5r$$

$$\Leftrightarrow \frac{(R+2r)^2(R-r)^2}{r^2}\left(\frac{4R+r}{r} + \frac{9r}{R-r} + 6\sqrt{\frac{4R+r}{R-r}}\right) \underbrace{\overset{?}{>}}_{(*)} 4R+5r$$

$$\text{Now, } \frac{4R+r}{R-r} \overset{?}{\geq} 9.\frac{4r^2}{R^2} \Leftrightarrow 4t^3+t^2-36t+36 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(4t^2+9(t-2)\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2$$

$$\therefore \text{LHS of } (*) > \frac{(R+2r)^2(R-r)^2}{r^2}\left(\frac{4R+r}{r} + \frac{9r}{R-r} + \frac{36r}{R}\right)$$

$$= \frac{(R+2r)^2(R-r)^2}{r^2}\left(\frac{R(4R+r)(R-r) + 9Rr^2 + 36r^2(R-r)}{Rr(R-r)}\right) \overset{?}{>} 4R+5r$$

$$\Leftrightarrow 4t^6+9t^5+35t^4+64t^3-56t^2-201t+144 \overset{?}{>} 0$$

$$\Leftrightarrow (t-2)\left(4t^5+17t^4+69t^3+202t^2+348t+495\right)+1134 \overset{?}{>} 0 \to \text{true}$$

$$\Rightarrow (*) \text{ is true} \Rightarrow \sum_{cyc}\sqrt{\frac{a}{b+c-a}} \leq \frac{R^2-4r^2}{r^2} + 3\sqrt{\frac{r}{R-r}} \overset{via\,(2)}{\leq} \sum_{cyc}\sqrt{\frac{m_a}{n_a+g_a-m_a}}$$

$$\therefore \sum_{cyc}\sqrt{\frac{m_a}{n_a+g_a-m_a}} + \frac{R^2}{r^2} \geq 4 + \sum_{cyc}\sqrt{\frac{a}{b+c-a}},$$

$'' ='' $ iff Δ ABC is equilateral (QED)