ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationships hold :

$$(i)\ \sum_{cyc} a^4 + abc(a+b+c) \ge \sum_{cyc} ab(a^2+b^2) + \frac{r^4(R-2r)}{16R}\ \text{and}$$

$$(ii)\ \frac{a+b+c}{3} + \frac{R^3-4r^3}{2r^2} \ge \sqrt[3]{\frac{a^3+b^3+c^3}{3}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} a^4 + abc(a+b+c) \ge \sum_{cyc} ab(a^2+b^2) + \frac{r^4(R-2r)}{16R}$$

$$\Leftrightarrow 2\sum_{cyc} a^2b^2 - 16r^2s^2 + abc\left(\sum_{cyc} a\right) \ge \sum_{cyc}\left(ab\left(\sum_{cyc} a^2 - c^2\right)\right) + \frac{r^4(R-2r)}{16R}$$

$$\Leftrightarrow 2\sum_{cyc} a^2b^2 - 16r^2s^2 + 2abc\left(\sum_{cyc} a\right) \ge \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right) + \frac{r^4(R-2r)}{16R}$$

$$\Leftrightarrow 2(s^2+4Rr+r^2)^2 - 32Rrs^2 - 16r^2s^2 + 2.4Rrs.2s$$

$$\ge 2(s^2-4Rr-r^2)(s^2+4Rr+r^2) + \frac{r^4(R-2r)}{16R}$$

$$\Leftrightarrow \frac{\begin{pmatrix} 32R(s^2+4Rr+r^2)^2 - 512R^2rs^2 - 256Rr^2s^2 + 256R^2rs^2 \\ -32R(s^2-4Rr-r^2)(s^2+4Rr+r^2) - r^4(R-2r) \end{pmatrix}}{16R} \ge 0$$

$$\Leftrightarrow 1024R^3 + 512R^2r + 63Rr^2 + 2r^3 \overset{(*)}{\ge} 192Rs^2$$

$$\text{Now, } 192Rs^2 \overset{\text{Gerretsen}}{\le} 192R(4R^2+4Rr+3r^2) \overset{?}{\le} 1024R^3 + 512R^2r + 63Rr^2$$

$$+2r^3 \Leftrightarrow 256t^3 - 256t^2 - 513t + 2 \overset{?}{\ge} 0\ \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(256t^2+256t-1) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore \boxed{\sum_{cyc} a^4 + abc(a+b+c) \ge \sum_{cyc} ab(a^2+b^2) + \frac{r^4(R-2r)}{16R}}$$

$$\forall\, \Delta ABC, '' = ''\ \text{iff } \Delta ABC \text{ is equilateral}$$

$$\text{Again, } \sqrt[3]{\frac{a^3+b^3+c^3}{3}} - \frac{a+b+c}{3} = \frac{\frac{a^3+b^3+c^3}{3} - \left(\frac{a+b+c}{3}\right)^3}{\left(\frac{\sum_{cyc} a^3}{3}\right)^{\frac{2}{3}} + \left(\frac{\sum_{cyc} a}{3}\right)^2 + \left(\frac{\sum_{cyc} a^3}{3}\right)^{\frac{1}{3}}\left(\frac{\sum_{cyc} a}{3}\right)}$$

$$\leq \frac{\frac{9\sum_{cyc} a^3 - \left(\sum_{cyc} a\right)^3}{27}}{\left(\frac{\left(\sum_{cyc} a\right)^3}{27}\right)^{\frac{2}{3}} + \left(\frac{\sum_{cyc} a}{3}\right)^2 + \left(\frac{\left(\sum_{cyc} a\right)^3}{27}\right)^{\frac{1}{3}}\left(\frac{\sum_{cyc} a}{3}\right)} = \frac{9\sum_{cyc} a^3 - \left(\sum_{cyc} a\right)^3}{9\left(\sum_{cyc} a\right)^2}$$

$$= \frac{18s(s^2 - 6Rr - 3r^2) - 8s^3}{9.4s^2} = \frac{1}{2}\cdot\frac{5s^2 - 54Rr - 27r^2}{9s} \overset{\text{Mitrinovic}}{\leq}$$

$$\frac{1}{2}\cdot\frac{5s^2 - 54Rr - 27r^2}{27\sqrt{3}r} \leq \frac{1}{2}\cdot\frac{5s^2 - 54Rr - 27r^2}{46r} \left(\begin{array}{c}\because 27\sqrt{3} > 46\ and \\ 5s^2 - 54Rr - 27r^2 \geq 0\end{array}\right)$$

$$\overset{\text{Gerretsen}}{\leq} \frac{1}{2}\cdot\frac{5(4R^2 + 4Rr + 3r^2) - 54Rr - 27r^2}{46r} \overset{?}{\leq} \frac{R^3 - 4r^3}{2r^2}$$

$$\Leftrightarrow 46(R^3 - 4r^3) \overset{?}{\geq} r\left(5(4R^2 + 4Rr + 3r^2) - 54Rr - 27r^2\right)$$

$$\Leftrightarrow 23t^3 - 10t^2 + 17t - 178 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(23t^2 + 36t + 89) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow \sqrt[3]{\frac{a^3 + b^3 + c^3}{3}} - \frac{a + b + c}{3} \leq \frac{R^3 - 4r^3}{2r^2}$$

$$\therefore \boxed{\frac{a + b + c}{3} + \frac{R^3 - 4r^3}{2r^2} \geq \sqrt[3]{\frac{a^3 + b^3 + c^3}{3}}}\ \forall\, \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$