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In any acute triangle ABC, the following relationship holds :

$$\frac{a}{h_a}\cdot\sqrt{\sin\frac{A}{2}}+\frac{b}{h_b}\cdot\sqrt{\sin\frac{B}{2}}+\frac{c}{h_c}\cdot\sqrt{\sin\frac{C}{2}}>\frac{2}{3}\cdot\sqrt{3}\cdot\sqrt[4]{8}$$

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$$\sum_{cyc} a\sin\frac{A}{2}=\sum_{cyc}\frac{a\left(2\cos\frac{B+c}{2}\cos\frac{B-c}{2}\right)}{2\cos\frac{B-c}{2}}\overset{0<\cos\frac{B-c}{2}\le 1 \text{ and analogs}}{\ge}$$

$$\sum_{cyc}\frac{a\left(2\cos\frac{B+c}{2}\cos\frac{B-c}{2}\right)}{2}=\sum_{cyc}\frac{a(\cos B+\cos C)}{2}=\frac{1}{2}\sum_{cyc}\left(a\left(\sum_{cyc}\cos A-\cos A\right)\right)$$

$$=\frac{1}{2}\left(\left(\sum_{cyc}\cos A\right)\left(\sum_{cyc}a\right)-\sum_{cyc}a\cos A\right)=\frac{1}{2}\left(\frac{2s(R+r)}{R}-R\sum_{cyc}\sin 2A\right)$$

$$=\frac{1}{2}\left(\frac{2s(R+r)}{R}-4R\prod_{cyc}\sin A\right)=\frac{1}{2}\left(\frac{2s(R+r)}{R}-\frac{4R.4Rrs}{8R^3}\right)$$

$$=\frac{1}{2}\left(\frac{2s(R+r)}{R}-\frac{2rs}{R}\right)=\frac{2Rs}{2R}=s\ \therefore\ \sum_{cyc}a\sin\frac{A}{2}\ge s\rightarrow(1)$$

$$\text{Now, }\frac{a}{h_a}\cdot\sqrt{\sin\frac{A}{2}}+\frac{b}{h_b}\cdot\sqrt{\sin\frac{B}{2}}+\frac{c}{h_c}\cdot\sqrt{\sin\frac{C}{2}}=\frac{1}{2rs}\cdot\sum_{cyc}\left(a^2\cdot\sqrt{\sin\frac{A}{2}}\right)$$

$$=\frac{1}{2rs}\cdot\sum_{cyc}\left(\frac{a^2.\sin\frac{A}{2}}{\sqrt{\sin\frac{A}{2}}}\right)\overset{0<\sin\frac{A}{2}\le\frac{1}{\sqrt{2}}\text{ and analogs as }\Delta\text{ ABC is acute}}{\ge}\frac{1}{2rs}\cdot\sum_{cyc}\left(\frac{a^2.\sin\frac{A}{2}}{\frac{1}{\sqrt[4]{2}}}\right)$$

$$\overset{\text{Chebyshev}}{\ge}\frac{\sqrt[4]{2}}{6rs}\cdot\left(\sum_{cyc}a\sin\frac{A}{2}\right)\left(\sum_{cyc}a\right)$$

$$\left(\begin{matrix}\because \text{WLOG assuming } a\ge b\ge c\Rightarrow\sin\frac{A}{2}\ge\sin\frac{B}{2}\ge\sin\frac{C}{2}\\ \Rightarrow a\sin\frac{A}{2}\ge b\sin\frac{B}{2}\ge c\sin\frac{C}{2}\end{matrix}\right)\overset{\text{via }(1)}{\ge}\frac{\sqrt[4]{2}}{6rs}\cdot(s)(2s)$$

$$=\frac{\sqrt[4]{2}.s}{3r}\overset{\text{Mitrinovic}}{\ge}\frac{\sqrt[4]{2}.3\sqrt{3}.r}{3r}=\sqrt[4]{2}.\sqrt{3}\overset{?}{>}\frac{2}{3}.\sqrt{3}.\sqrt[4]{8}\Leftrightarrow 3\overset{?}{>}2\sqrt{2}\Leftrightarrow 9\overset{?}{>}8\rightarrow\text{true}$$

$$\therefore\frac{a}{h_a}\cdot\sqrt{\sin\frac{A}{2}}+\frac{b}{h_b}\cdot\sqrt{\sin\frac{B}{2}}+\frac{c}{h_c}\cdot\sqrt{\sin\frac{C}{2}}>\frac{2}{3}\cdot\sqrt{3}.\sqrt[4]{8}\ \forall\text{ acute }\Delta\text{ ABC (QED)}$$