ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC, the following relationship holds :

$$\frac{a}{h_a}\sqrt{\tan A}+\frac{b}{h_b}\sqrt{\tan B}+\frac{c}{h_c}\sqrt{\tan C}>2\sqrt[4]{27}$$

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Let $f(x)=(\sin x)\sqrt{\tan x}\ \forall x\in\left(0,\frac{\pi}{2}\right)$ and then : $f''(x)$

$$=\frac{(4\sec^2 x-4)(\sin x)(\tan^2 x)+4(\sin x)(\sec^2 x)-(\sec^4 x)(\sin x)}{4\tan^{\frac{3}{2}} x}$$

$$=\frac{4(\sin x)(\tan^4 x)+3(\sin x)(\sec^2 x)-(\sin x)(\sec^2 x)(\sec^2 x-1)}{4\tan^{\frac{3}{2}} x}$$

$$=\frac{4(\sin x)(\tan^4 x)+3\sin x+3(\sin x)(\tan^2 x)-(\sin x)(\sec^2 x)(\tan^2 x)}{4\tan^{\frac{3}{2}} x}$$

$$=\frac{4(\sin x)(\tan^4 x)+3\sin x+3(\sin x)(\tan^2 x)-(\sin x)(\tan^2 x)-(\sin x)(\tan^4 x)}{4\tan^{\frac{3}{2}} x}$$

$$=\frac{3(\sin x)(\tan^4 x)+3\sin x+2(\sin x)(\tan^2 x)}{4\tan^{\frac{3}{2}} x}>0\Rightarrow f(x)\text{ is convex}\rightarrow(1)$$

Now, WLOG if we assume $a\ge b\ge c$, then : $a\sqrt{\tan A}\ge b\sqrt{\tan B}\ge c\sqrt{\tan C}$

$(\because \Delta\text{ ABC is acute})$ and $\frac{1}{h_a}\ge\frac{1}{h_b}\ge\frac{1}{h_c}$ $\therefore$ via Chebyshev,

$$\frac{a}{h_a}\sqrt{\tan A}+\frac{b}{h_b}\sqrt{\tan B}+\frac{c}{h_c}\sqrt{\tan C}\ge\frac{1}{3}\left(\sum_{cyc}\frac{1}{h_a}\right)\left(\sum_{cyc}a\sqrt{\tan A}\right)\overset{\text{Jensen,via (1)}}{\ge}$$

$$\frac{3.2R}{3r}.\sin\frac{\pi}{3}.\sqrt{\tan\frac{\pi}{3}}=\frac{2R}{r}.\frac{\sqrt{3}}{2}.\sqrt{\sqrt{3}}\overset{\text{Euler}}{\ge}2.3^{\frac{3}{4}}=2\sqrt[4]{27}$$

$\forall$ acute Δ ABC, '' = '' iff Δ ABC is equilateral (QED)