ROMANIAN MATHEMATICAL MAGAZINE

In any acute triangle ABC, the following relationship holds :

$$(p-a)\sqrt{\cot A}+(p-b)\sqrt{\cot B}+(p-c)\sqrt{\cot C}>\frac{2}{3}p$$

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WLOG we may assume $a\ge b\ge c$ and then : $p-a\le p-b\le p-c$ and $\sqrt{\cot A}\le\sqrt{\cot B}\le\sqrt{\cot C}\therefore(p-a)\sqrt{\cot A}+(p-b)\sqrt{\cot B}+(p-c)\sqrt{\cot C}$

$$\overset{\text{Chebyshev}}{\ge}\frac{1}{3}\left(\sum_{cyc}(p-a)\right)\left(\sum_{cyc}\sqrt{\cot A}\right)\overset{?}{>}\frac{2}{3}p\Leftrightarrow\sum_{cyc}\sqrt{\cot A}\overset{?}{>}2\to(1)$$

Let $\sqrt{\cot A}=x,\sqrt{\cot B}=y,\sqrt{\cot C}=z\therefore(1)\Leftrightarrow\sum_{cyc}x>2\Leftrightarrow\left(\sum_{cyc}x\right)^2>4=$

$$4.\sqrt{\sum_{cyc}x^2y^2}\left(\because\sum_{cyc}\cot A\cot B=\sum_{cyc}x^2y^2=1\right)\Leftrightarrow\left(\sum_{cyc}x\right)^4>16\sum_{cyc}x^2y^2\to(2)$$

Assigning $y+z=X,z+x=Y,x+y=Z\Rightarrow X+Y-Z=2z>0,Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z,Y+Z>X,Z+X>Y\Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\to(\text{i})\Rightarrow x=s-X,y=s-Y,$ $z=s-Z$ and such substitutions $\Rightarrow\sum_{cyc}xy=\sum_{cyc}(s-X)(s-Y)$

$$\Rightarrow\sum_{cyc}xy=4Rr+r^2\to(\text{ii})\text{ and }\sum_{cyc}x^2y^2=\left(\sum_{cyc}xy\right)^2-2xyz\left(\sum_{cyc}x\right)$$

$$\overset{\text{via (i) and (ii)}}{=}(4Rr+r^2)^2-2\left(\prod_{cyc}(s-X)\right).s=(4Rr+r^2)^2-2r^2s.s$$

$$\Rightarrow\sum_{cyc}x^2y^2=r^2\left((4R+r)^2-2s^2\right)\to(\text{iii})\therefore\text{via (i) and (iii)},(2)\Leftrightarrow$$

$$s^4>16r^2\left((4R+r)^2-2s^2\right)\Leftrightarrow s^4+32r^2s^2\overset{(*)}{>}16r^2(4R+r)^2$$

Now, LHS of $(*)\overset{\text{Gerretsen}}{\ge}(16Rr+27r^2)s^2\overset{\text{Gerretsen}}{\ge}(16Rr+27r^2)(16Rr-5r^2)$ $\overset{?}{>}16r^2(4R+r)^2\Leftrightarrow76r(R-2r)+148Rr+r^2\overset{?}{>}0\to$ true $\because R\overset{\text{Euler}}{\ge}2r$

$\Rightarrow(*)\Rightarrow(2)\Rightarrow(1)$ is true

$$\therefore(p-a)\sqrt{\cot A}+(p-b)\sqrt{\cot B}+(p-c)\sqrt{\cot C}>\frac{2}{3}p\ \forall\text{ acute }\Delta\,ABC\ (QED)$$