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In any acute triangle ABC, the following relationship holds :

$$\frac{a}{b+c}.\sqrt{\sin A}+\frac{b}{c+a}.\sqrt{\sin B}+\frac{c}{a+b}.\sqrt{\sin C}>1$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Via Power}-\text{Mean}-\text{Inequality},\left(\frac{\sum_{cyc}a^{\frac{3}{2}}}{3}\right)^{\frac{2}{3}}\geq\left(\frac{\sum_{cyc}a^{1}}{3}\right)^{1}$$

$$\Rightarrow\frac{\sum_{cyc}a^{\frac{3}{2}}}{3}\geq\left(\frac{2s}{3}\right)^{\frac{3}{2}}=\frac{2s}{3}.\sqrt{\frac{2s}{3}}\Rightarrow\sum_{cyc}a^{\frac{3}{2}}\geq 2s.\sqrt{\frac{2s}{3}}\rightarrow(1)$$

$$\frac{a}{b+c}.\sqrt{\sin A}+\frac{b}{c+a}.\sqrt{\sin B}+\frac{c}{a+b}.\sqrt{\sin C}=\frac{1}{\sqrt{2R}}.\sum_{cyc}\frac{a\sqrt{a}}{b+c}\overset{\text{Chebyshev}}{\geq}$$

$$\frac{1}{3.\sqrt{2R}}.\left(\sum_{cyc}a\sqrt{a}\right)\left(\frac{1}{b+c}\right)\left(\begin{matrix}\because\text{WLOG assuming }a\geq b\geq c\Rightarrow a\sqrt{a}\geq b\sqrt{b}\geq c\sqrt{c}\\ \text{and }\frac{1}{b+c}\geq\frac{1}{c+a}\geq\frac{1}{a+b}\end{matrix}\right)\overset{\substack{\text{via (1)}\\ \text{and}\\ \text{Bergstrom}}}{\geq}$$

$$\frac{1}{3.\sqrt{2R}}.2s.\sqrt{\frac{2s}{3}}.\frac{9}{4s}=\frac{3}{2.\sqrt{2R}}.\sqrt{\frac{2s}{3}}>\frac{3}{2.\sqrt{2R}}.\sqrt{\frac{4R}{3}}$$

$$(\because\Delta\text{ ABC is acute}\Rightarrow s>2R+r>2R)=\sqrt{\frac{3}{2}}>1$$

$$\therefore\frac{a}{b+c}.\sqrt{\sin A}+\frac{b}{c+a}.\sqrt{\sin B}+\frac{c}{a+b}.\sqrt{\sin C}>1\ \forall\text{ acute }\Delta\text{ ABC (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we assume that $a\geq b\geq c$.

$$\text{We have }\frac{a}{b+c}\geq\frac{b}{c+a}\geq\frac{c}{a+b}\ \&\ \sqrt{\sin A}\geq\sqrt{\sin B}\geq\sqrt{\sin C},$$

then by Chebyshev's inequality, we have

$$\frac{a}{b+c}\sqrt{\sin A}+\frac{b}{c+a}\sqrt{\sin B}+\frac{c}{a+b}\sqrt{\sin C}\geq$$
$$\geq\frac{1}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\left(\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}\right).$$

By Nesbitt's inequality, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Also since $0<\sin x<1,\forall x\in\left(0,\frac{\pi}{2}\right)$, and by using Jordan's inequality, we have

$$\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}>\sin A+\sin B+\sin C\geq\frac{2A}{\pi}+\frac{2B}{\pi}+\frac{2C}{\pi}=2.$$

Therefore

$$\frac{a}{b+c}\sqrt{\sin A}+\frac{b}{c+a}\sqrt{\sin B}+\frac{c}{a+b}\sqrt{\sin C}>\frac{1}{3}.\frac{3}{2}.2=1.$$