ROMANIAN MATHEMATICAL MAGAZINE

In any acute triangle ABC holds:

$$\frac{1}{h_a}\sqrt{\tan A}+\frac{1}{h_b}\sqrt{\tan B}+\frac{1}{h_c}\sqrt{\tan C}\ge \frac{2}{R}\sqrt[4]{3}$$

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Note 1: $A+B+C=\pi$

$$\therefore A+B=\pi-C$$

$$\tan(A+B)=\tan(\pi-C)\Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C\Rightarrow \sum \tan A$$
$$=\tan A\cdot\tan B\cdot\tan C$$

Note 2: In any acute triangle : $\tan A+\tan B+\tan C\ge 3\sqrt{3}$

Proof: Since $f(x)=\tan x$ is convex on $\left(0,\frac{\pi}{2}\right)$. So by Jensen's inequality

$$\tan A+\tan B+\tan C\ge 3\tan\frac{A+B+C}{3}=3\tan\frac{\pi}{3}=3\sqrt{3}$$

$$\therefore \sum\frac{1}{h_a}\sqrt{\tan A}\overset{Chebyshev}{\ge}\frac{1}{3}\cdot\sum\frac{1}{h_a}\cdot\sum\sqrt{\tan A}$$

[WLOG $a\ge b\ge c$ $\therefore h_a\le h_b\le h_c\Rightarrow \frac{1}{h_a}\ge\frac{1}{h_b}\ge\frac{1}{h_c}$; $\tan A\ge\tan B\ge\tan C$]

$$\therefore \sum\frac{1}{h_a}\sqrt{\tan A}\ge\frac{1}{3}\cdot\frac{1}{r}\cdot\sum\sqrt{\tan A}\overset{AM-GM}{\ge}\frac{1}{3r}\cdot 3(\tan A\tan B\tan C)^{\frac{1}{6}}=$$

$$=\frac{1}{r}\left(3\sqrt{3}\right)^{\frac{1}{6}}\overset{Euler}{\ge}\frac{2}{R}\cdot\left(3^{\frac{3}{2}}\right)^{\frac{1}{6}}=\frac{2}{R}\sqrt[4]{3}$$