ROMANIAN MATHEMATICAL MAGAZINE

In any acute triangle ABC, the following relationship holds :

$$\frac{a}{b+c}\sqrt{\cos A}+\frac{b}{c+a}\sqrt{\cos B}+\frac{c}{a+b}\sqrt{\cos C}<\sqrt{2}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$a\cos A+b\cos B+c\cos C=R(\sin 2A+\sin 2B+\sin 2C)$$
$$=R(2\sin A\cos A+2\sin A\cos(B-C))=2R\sin A\,(\cos(B-C)-\cos(B+C))$$
$$=4R\sin A\sin B\sin C=4R.\frac{4Rrs}{8R^3}\Rightarrow a\cos A+b\cos B+c\cos C=\frac{2rs}{R}\rightarrow(1)$$
$$\text{Now, } b+c=2s-a=s+s-a>s\Rightarrow b+c>s \text{ and analogs}\rightarrow(2)$$
$$\therefore\frac{a}{b+c}\sqrt{\cos A}+\frac{b}{c+a}\sqrt{\cos B}+\frac{c}{a+b}\sqrt{\cos C}\overset{\text{via }(2)}{<}\frac{1}{s}\sum_{\text{cyc}}\left(\sqrt{a\cos A}.\sqrt{a}\right)$$
$$\overset{\text{CBS}}{\leq}\frac{1}{s}.\sqrt{\sum_{\text{cyc}}a\cos A}.\sqrt{\sum_{\text{cyc}}a}\overset{\text{via }(1)}{=}\frac{1}{s}.\sqrt{\frac{2rs}{R}}.\sqrt{2s}=\sqrt{2}.\sqrt{\frac{2r}{R}}\overset{\text{Euler}}{\leq}\sqrt{2}$$
$$\therefore\frac{a}{b+c}\sqrt{\cos A}+\frac{b}{c+a}\sqrt{\cos B}+\frac{c}{a+b}\sqrt{\cos C}<\sqrt{2}\ \forall \text{ acute }\Delta\,ABC\ (QED)$$

Solution 2 by Tapas Das-India

$$\text{We know that in any triangle } a+b>c \text{ or } a+b>\frac{1}{2}(a+b+c),$$
$$\text{similarly } b+c>\frac{1}{2}(a+b+c) \text{ and } c+a>\frac{1}{2}(a+b+c).$$
$$\text{Now we have}\sum\frac{a}{b+c}<\frac{2(a+b+c)}{a+b+c}=2\,.$$

Back to the main problem,

$$LHS\overset{\text{Chebyshev}}{\leq}\frac{1}{3}\sum\frac{a}{b+c}\sum\sqrt{\cos A}\overset{CBS}{\leq}\frac{1}{3}.2.\sqrt{3\sum\cos A}<\frac{2}{3}\sqrt{\frac{9}{2}}=\sqrt{2}$$
$$\text{Note:}\sum\cos A=1+\frac{r}{R}\leq\frac{3}{2}(\text{Euler}) \text{ and WLOG } a\geq b\geq c,$$
$$\text{so }\cos A\leq\cos B\leq\cos C \text{ and }\frac{a}{b+c}\geq\frac{b}{c+a}\geq\frac{c}{a+b}$$