ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{w_a^5 m_b^5}{h_b^2 r_c^2}+\frac{w_b^5 m_c^5}{h_c^2 r_a^2}+\frac{w_c^5 m_a^5}{h_a^2 r_b^2}\ge\frac{27\cdot 6^4\cdot r^{10}}{R^4}$$

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$$\sum_{cyc}\frac{r_c}{h_a}=\sum_{cyc}\left(\frac{rs}{s-c}\cdot\frac{a}{2rs}\right)=\frac{1}{2r^2s}\sum_{cyc}a(s-a)(s-b)$$

$$\overset{A-G}{\le}\frac{1}{8r^2s}\sum_{cyc}a(2s-a-b)^2=\frac{1}{8r^2s}\sum_{cyc}ac^2\overset{A-G}{\le}\frac{1}{8r^2s}\sum_{cyc}a^3=\frac{2s(s^2-6Rr-3r^2)}{8r^2s}$$

$$\therefore\sum_{cyc}\frac{r_c}{h_a}\overset{(i)}{\le}\frac{s^2-6Rr-3r^2}{4r^2}$$

$$\sum_{cyc}w_aw_b\ge\sum_{cyc}h_ah_b=\sum_{cyc}\frac{bc.ca}{4R^2}=\frac{rs}{R}\sum_{cyc}a=\frac{2rs^2}{R}\Rightarrow\sum_{cyc}w_aw_b\overset{(ii)}{\ge}\frac{2rs^2}{R}$$

$$\text{Now, }\frac{w_a^5 m_b^5}{h_b^2 r_c^2}+\frac{w_b^5 m_c^5}{h_c^2 r_a^2}+\frac{w_c^5 m_a^5}{h_a^2 r_b^2}=\sum_{cyc}\frac{(w_am_b)^3}{\left(\frac{h_br_c}{w_am_b}\right)^2}\ge\sum_{cyc}\frac{(w_aw_b)^3}{\left(\frac{r_c}{h_a}\right)^2}\overset{Radon}{\ge}\frac{\left(\sum_{cyc}w_aw_b\right)^3}{\left(\sum_{cyc}\frac{r_c}{h_a}\right)^2}$$

$$\overset{\text{via (i),(ii)}}{\ge}\frac{8r^3s^6*16r^4}{R^3(s^2-6Rr-3r^2)^2}\overset{Gerretsen}{\ge}\frac{4r^3*16r^4s^4*\left(27Rr+5r(R-2r)\right)}{R^3(s^2-6Rr-3r^2)^2}$$

$$\overset{Euler}{\ge}\frac{4r^3*16r^4s^4*27Rr}{R^3(s^2-6Rr-3r^2)^2}\overset{?}{\ge}\frac{27*6^4*r^{10}}{R^4}=\frac{27*81*16*r^{10}}{R^4}$$

$$\Leftrightarrow 2Rs^2\overset{?}{\ge}9r\left(s^2-6Rr-3r^2\right)\Leftrightarrow(2R-9r)s^2+9r\left(6Rr+3r^2\right)\underset{(*)}{\overset{?}{\ge}}0$$

Case 1: $2R-9r\ge 0$ *and then* : LHS of $(*)\ge 9r\left(6Rr+3r^2\right)>0$

$\Rightarrow(*)$ is true (strict inequality)

Case 2: $2R-9r<0$ *and then* : LHS of $(*)=-(9r-2R)s^2+9r\left(6Rr+3r^2\right)$

$$\overset{Gerretsen}{\ge}-(9r-2R)\left(4R^2+4Rr+3r^2\right)+9r\left(6Rr+3r^2\right)\overset{?}{\ge}0$$

$$\Leftrightarrow 4R\left(2R^2-7Rr+6r^2\right)\overset{?}{\ge}0\Leftrightarrow 4R(2R-3r)(R-2r)\overset{?}{\ge}0\rightarrow\text{true}\because R\overset{Euler}{\ge}2r$$

$\Rightarrow(*)$ is true $\therefore$ combining cases 1 *and* 2, $(*)$ is true $\forall\,\Delta$ ABC

$$\therefore\frac{w_a^5 m_b^5}{h_b^2 r_c^2}+\frac{w_b^5 m_c^5}{h_c^2 r_a^2}+\frac{w_c^5 m_a^5}{h_a^2 r_b^2}\ge\frac{27*6^4*r^{10}}{R^4}$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)