ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$① \left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^4+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^4+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^4 \ge \frac{3.2^{12}.r^8}{3(3R^4-32r^4)^2-512r^8}$$

$$② \left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^5+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^5+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^5 \ge \frac{3.2^{15}.r^{10}}{3(81R^5-2560r^5)^2-2^{11}r^{10}}$$

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Proof of ① :

$$\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^4+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^4+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^4$$

$$\ge \left(\frac{h_a}{w_b}+\frac{w_c}{h_a}\right)^4+\left(\frac{h_b}{w_c}+\frac{w_a}{h_b}\right)^4+\left(\frac{h_c}{w_a}+\frac{w_b}{h_c}\right)^4 \overset{A-G}{\ge} 16\left(\left(\frac{w_c}{w_b}\right)^2+\left(\frac{w_a}{w_c}\right)^2+\left(\frac{w_b}{w_a}\right)^2\right)$$

$$\ge 16\left(\left(\frac{w_c}{w_b}\right)\left(\frac{w_a}{w_c}\right)+\left(\frac{w_a}{w_c}\right)\left(\frac{w_b}{w_a}\right)+\left(\frac{w_b}{w_a}\right)\left(\frac{w_c}{w_b}\right)\right)=16\left(\frac{w_a}{w_b}+\frac{w_b}{w_c}+\frac{w_c}{w_a}\right)$$

$$=16\left(\frac{w_a^2}{w_aw_b}+\frac{w_b^2}{w_bw_c}+\frac{w_c^2}{w_cw_a}\right) \underset{\text{and } w_a\le\sqrt{s(s-a)} \text{ and analogs}}{\overset{\text{Bergstrom } + w_a \ge h_a \text{ and analogs}}{\ge}}$$

$$16.\frac{\left(\sum_{cyc}\frac{2rs}{a}\right)^2}{\sum_{cyc}\left(\sqrt{s(s-a)}\sqrt{s(s-b)}\right)} \overset{\text{Bergstrom| and CBS}}{\ge} \frac{16\left(\frac{2rs.9}{2s}\right)^2}{s.\sqrt{3s-2s}.\sqrt{3s-2s}}=\frac{16.81r^2}{s^2} \overset{\text{Mitrinovic}}{\ge}$$

$$\frac{2^6.81r^2}{27R^2} \overset{?}{\ge} \frac{3.2^{12}.r^8}{3(3R^4-32r^4)^2-512r^8} \Leftrightarrow 3\left(3R^4-32r^4\right)^2 \underset{(*)}{\overset{?}{\ge}} 512r^8+64R^2r^6$$

$$\text{Now, } 512r^8+64R^2r^6 \overset{\text{Euler}}{\le} 128R^2r^6+64R^2r^6=3.64R^2r^6 \overset{?}{\le} 3\left(3R^4-32r^4\right)^2$$

$$\Leftrightarrow 3R^4-32r^4 \overset{?}{\ge} 8Rr^3 \Leftrightarrow 3R\left(R^3-8r^3\right)+16r^3(R-2r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r$$

$$\therefore \left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^4+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^4+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^4 \ge \frac{3.2^{12}.r^8}{3(3R^4-32r^4)^2-512r^8}$$

Proof of ② :

$$\text{Now, } \left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^3+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^3+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^3 \overset{A-G}{\ge}$$

$$3\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right) \overset{A-G}{\ge} 3.8.\sqrt{\frac{m_a}{w_b}.\frac{w_c}{h_a}.\frac{m_b}{w_c}.\frac{w_a}{h_b}.\frac{m_c}{w_a}.\frac{w_b}{h_c}} \ge 3.8 \overset{?}{\ge}$$

$$\frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6} \Leftrightarrow \left(9R^3-64r^3\right)^2 \overset{?}{\ge} 64r^6 \Leftrightarrow 9R^3-64r^3 \overset{?}{\ge} 8r^3$$

$$\Leftrightarrow R^3 \overset{?}{\ge} 8r^3 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r \therefore \left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^3+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^3+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^3$$

$$\geq \frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6}\ \forall\ \Delta\ ABC \to (1)$$

$$\text{Via Chebyshev, } \left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^5+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^5+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^5 \geq$$

$$\frac{1}{3}\left(\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^2+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^2+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^2\right)\left(\begin{array}{c}\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^3+\\ \left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^3+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^3\end{array}\right)$$

$$\overset{\text{via }(1)}{\geq}\frac{1}{3}\left(\left(\frac{h_a}{w_b}+\frac{w_c}{h_a}\right)^2+\left(\frac{h_b}{w_c}+\frac{w_a}{h_b}\right)^2+\left(\frac{h_c}{w_a}+\frac{w_b}{h_c}\right)^2\right).\frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6}\overset{A-G}{\geq}$$

$$\frac{4}{3}\left(\frac{w_c}{w_b}+\frac{w_a}{w_c}+\frac{w_b}{w_a}\right).\frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6}\overset{A-G}{\geq}\frac{4}{3}.3.\frac{3.2^9.r^6}{3(9R^3-64r^3)^2-128r^6}$$

$$\overset{?}{\geq}\frac{3.2^{15}.r^{10}}{3(81R^5-2560r^5)^2-2^{11}r^{10}}$$

$$\Leftrightarrow 3\left(81R^5-2560r^5\right)^2-2^{11}r^{10}\overset{?}{\geq}48r^4\left(9R^3-64r^3\right)^2-2^4.r^4.2^7.r^6$$

$$\Leftrightarrow 81R^5-2560r^5\overset{?}{\geq}4r^2\left(9R^3-64r^3\right)\Leftrightarrow 9R^5-4R^3r^2-256r^5\overset{?}{\geq}0$$

$$\Leftrightarrow R^3\left(R^2-4r^2\right)+8\left(R^5-32r^5\right)\overset{?}{\geq}0\to \text{true}\because R\overset{\text{Euler}}{\geq}2r$$

$$\therefore \left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^5+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^5+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^5\geq\frac{3.2^{15}.r^{10}}{3(81R^5-2560r^5)^2-2^{11}r^{10}}$$

$$\therefore (1)\ \left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^4+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^4+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^4\geq\frac{3.2^{12}.r^8}{3(3R^4-32r^4)^2-512r^8}\ \textit{and}$$

$$(2)\ \left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^5+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^5+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^5\geq\frac{3.2^{15}.r^{10}}{3(81R^5-2560r^5)^2-2^{11}r^{10}}$$

$\forall\ \Delta\ ABC$, **equalities iff Δ ABC is equilateral (QED)**