ROMANIAN MATHEMATICAL MAGAZINE

If $n \in \mathbb{N}$ then in ΔABC the following relationship holds:

$$\frac{h_a^n}{\sin A}+\frac{h_b^n}{\sin B}+\frac{h_c^n}{\sin C}\geq 2\cdot 3^{\frac{2n+1}{2}}\cdot r^n$$

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$$\frac{h_a^n}{\sin A}+\frac{h_b^n}{\sin B}+\frac{h_c^n}{\sin C}=\sum_{cyc}\frac{h_a^n}{\sin A}=\sum_{cyc}\frac{\left(\frac{2F}{a}\right)^n}{\sin A}=$$

$$=(2F)^n\cdot\sum_{cyc}\frac{1}{a^n\sin A}=(2rs)^n\cdot\sum_{cyc}\frac{1}{a^n\cdot\frac{a}{2R}}=$$

$$=2^{n+1}\cdot r^n\cdot s^n\cdot R\sum_{cyc}\frac{1^{n+2}}{a^{n+1}}\overset{RADON}{\geq}2^{n+1}\cdot r^n\cdot s^n\cdot R\cdot\frac{(1+1+1)^{n+2}}{(a+b+c)^{n+1}}=$$

$$=2^{n+1}\cdot r^n\cdot s^n\cdot R\cdot\frac{(3)^{n+2}}{(2s)^{n+1}}=r^n\cdot R\cdot\frac{3^{n+2}}{s}\overset{MITRINOVIC}{\geq}$$

$$\geq r^n\cdot R\cdot\frac{3^{n+2}}{\frac{3\sqrt{3}}{2}\cdot R}=2\cdot 3^{n+2-\frac{3}{2}}\cdot r^n=2\cdot 3^{\frac{2n+1}{2}}\cdot r^n$$

Equality holds for $a=b=c$.