ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{h_a^n}{\sin\frac{A}{2}}+\frac{h_b^n}{\sin\frac{B}{2}}+\frac{h_c^n}{\sin\frac{C}{2}}\ge 2\,.\,3^{n+1}\,r^n, n\in N$$

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Solution by Tapas Das-India

$$WLOG\ a\ge b\ge c\ then\ \sin\frac{A}{2}\ge\sin\frac{B}{2}\ge\sin\frac{C}{2}\ and$$

$$h_a\le h_b\le h_c\ and\ h_a^n\le h_b^n\le h_c^n, \frac{1}{\sin\frac{A}{2}}\le\frac{1}{\sin\frac{B}{2}}\le\frac{1}{\sin\frac{C}{2}}$$

$$\sum h_a^n=(2F)^n\sum\frac{1}{a^n}=(2rs)^n\sum\frac{(1)^{n+1}}{a^n}\overset{Radon}{\ge}$$

$$\ge\frac{(2rs)^n3^{n+1}}{(a+b+c)^n}=\frac{(2rs)^n3^{n+1}}{(2s)^n}=r^n3^{n+1}\ (1)$$

$$\sum\frac{1}{\sin\frac{A}{2}}\overset{AM-GM}{\ge}3\left(\frac{1}{\prod\sin\frac{A}{2}}\right)^{\frac{1}{3}}=3\left(\frac{4R}{r}\right)^{\frac{1}{3}}\overset{Euler}{\ge}6\ (2)$$

$$\frac{h_a^n}{\sin\frac{A}{2}}+\frac{h_b^n}{\sin\frac{B}{2}}+\frac{h_c^n}{\sin\frac{C}{2}}\overset{Chebyshev}{\ge}\frac{1}{3}\left(\sum h_a^n\right)\left(\sum\frac{1}{\sin\frac{A}{2}}\right)\overset{(1)\&(2)}{\ge}$$

$$\ge\frac{1}{3}\,r^n3^{n+1}.6=2\,.\,3^{n+1}\,r^n$$

$$Equality\ for\ a=b=c$$