ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a^n}{\cos\frac{A}{2}}+\frac{b^n}{\cos\frac{B}{2}}+\frac{c^n}{\cos\frac{C}{2}}\ge 2^{n+1}\,3^{\frac{n+1}{2}}\,r^n,\qquad \forall\, n\in N$$

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• $for\ n=1\ we\ need\ to\ show\ \frac{a}{\cos\frac{A}{2}}+\frac{b}{\cos\frac{B}{2}}+\frac{c}{\cos\frac{C}{2}}\ge 12r$,

$$\frac{a}{\cos\frac{A}{2}}+\frac{b}{\cos\frac{B}{2}}+\frac{c}{\cos\frac{C}{2}}=4R\sum\sin\frac{A}{2}\overset{am-gm}{\ge}12R\left(\frac{r}{4R}\right)^{\frac{1}{3}}=$$

$$=12R\left(\frac{r^3}{4Rr^2}\right)^{\frac{1}{3}}\overset{Euler}{\ge}12R.\frac{r}{R}=12r$$

• $for\ n=2, we\ need\ to\ show\ \frac{a^2}{\cos\frac{A}{2}}+\frac{b^2}{\cos\frac{B}{2}}+\frac{c^2}{\cos\frac{C}{2}}\ge 24\sqrt{3}\,r^2$

$$\frac{a^2}{\cos\frac{A}{2}}+\frac{b^2}{\cos\frac{B}{2}}+\frac{c^2}{\cos\frac{C}{2}}\ge\frac{(a+b+c)^2}{\sum\cos\frac{A}{2}}\overset{Jensen}{\ge}$$

$$\ge\frac{4s^2}{3\cos\frac{\pi}{6}}\overset{Mitrinovic}{\ge}4.27r^2,\frac{2}{3\sqrt{3}}=24\sqrt{3}\,r^2$$

• $for\ n>2\ we\ need\ to\ show\ \frac{a^n}{\cos\frac{A}{2}}+\frac{b^n}{\cos\frac{B}{2}}+\frac{c^n}{\cos\frac{C}{2}}\ge 2^{n+1}\,3^{\frac{n+1}{2}}\,r^n$

$$\left(\frac{a^n}{\cos\frac{A}{2}}+\frac{b^n}{\cos\frac{B}{2}}+\frac{c^n}{\cos\frac{C}{2}}\right)\left(\sum\cos\frac{A}{2}\right)(1+1+1)^{n-2}\overset{Holder}{\ge}(a+b+c)^n$$

$$\left(\frac{a^n}{\cos\frac{A}{2}}+\frac{b^n}{\cos\frac{B}{2}}+\frac{c^n}{\cos\frac{C}{2}}\right)\ge\frac{(2s)^n}{\left(\sum\cos\frac{A}{2}\right)(1+1+1)^{n-2}}\overset{Jensen}{\ge}$$

$$\ge\frac{2^ns^n}{3\cos\left(\frac{\pi}{6}\right).3^{n-2}}\overset{Mitrinovic}{\ge}\frac{2^n3^{\frac{3n}{2}}r^n}{\frac{3\sqrt{3}}{2}.3^{n-2}}=2^{n+1}\,3^{\frac{n+1}{2}}\,r^n$$

Equality for $a=b=c$