ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a^n}{\sin^2 A}+\frac{b^n}{\sin^2 B}+\frac{c^n}{\sin^2 C}\ge 2^{n+2}\cdot 3^{\frac{n}{2}}\cdot r^n,\qquad n\in N$$

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$$abc=4Rrs\le 4R\frac{R}{2}\frac{3\sqrt{3}R}{2}(Euler\ \&\ Mitrinovic)=R^3\left(\sqrt{3}\right)^3(1)$$

$$and\ \sum\sin^2 A=\frac{a^2+b^2+c^2}{4R^2}\overset{Leibniz}{\le}\frac{9}{4}(2)$$

• $for\ n=1\ ,we\ need\ to\ show\ \frac{a}{\sin^2 A}+\frac{b}{\sin^2 B}+\frac{c}{\sin^2 C}\ge 8\sqrt{3}r$

$$\frac{a}{\sin^2 A}+\frac{b}{\sin^2 B}+\frac{c}{\sin^2 C}=2R\sum\frac{1}{\sin A}=4R^2\sum\frac{1}{a}\overset{Am-Gm}{\ge}$$
$$\ge\ 12R^2\left(\frac{1}{abc}\right)^{\frac{1}{3}}\overset{(1)}{\ge}\frac{12R^2}{R\sqrt{3}}=8\sqrt{3}r\ (Euler)$$

• $for\ n=2\ we\ need\ to\ show\ \frac{a^2}{\sin^2 A}+\frac{b^2}{\sin^2 B}+\frac{c^2}{\sin^2 C}\ge 48r^2$

$$\frac{a^2}{\sin^2 A}+\frac{b^2}{\sin^2 B}+\frac{c^2}{\sin^2 C}=12R^2\ge 48r^2(Euler)$$

• $for\ \ n>2\ we\ need\ to\ show\ \frac{a^n}{\sin^2 A}+\frac{b^n}{\sin^2 B}+\frac{c^n}{\sin^2 C}\ge 2^{n+2}3^{\frac{n}{2}}r^n,n\in N$

$$\left(\frac{a^n}{\sin^2 A}+\frac{b^n}{\sin^2 B}+\frac{c^n}{\sin^2 C}\right)\left(\sum\sin^2 A\right)(1+1+1)^{n-2}\overset{Holder}{\ge}(a+b+c)^n$$

$$\left(\frac{a^n}{\sin^2 A}+\frac{b^n}{\sin^2 B}+\frac{c^n}{\sin^2 C}\right)\overset{(2)}{\ge}2^n s^n.\frac{4}{9.3^{n-2}}\overset{Mitrinovic}{\ge}2^n\,3^{\frac{3n}{2}}.\frac{4r^n}{9.3^{n-2}}=2^{n+2}3^{\frac{n}{2}}r^n$$

• $Equality\ for\ a=b=c$