ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\left(\frac{m_a}{w_b}\right)^2+\left(\frac{w_b}{h_c}\right)^2+\left(\frac{h_c}{m_a}\right)^2\le 9.\left(\frac{9}{8}.\left(\frac{R}{r}\right)^3-8\right)^2-6$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{m_a}{h_b}\right)^2+\left(\frac{m_b}{h_c}\right)^2+\left(\frac{m_c}{h_a}\right)^2=\sum_{cyc}\frac{2b^2+2c^2+2a^2-3a^2}{4h_b^2}$$

$$=\frac{1}{2}\left(\sum_{cyc}a^2\right)\left(\frac{1}{4r^2s^2}\right)\left(\sum_{cyc}a^2\right)-\frac{3}{4.4r^2s^2}\sum_{cyc}a^2b^2\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Gerretsen}}}{\le}$$

$$\frac{81R^4}{4r^2.(27Rr+5r(R-2r))}-\frac{3abc(a+b+c)}{16r^2s^2}\overset{\text{Euler}}{\le}\frac{81R^4}{4r^2.27Rr}-\frac{3.4Rrs.2s}{16r^2s^2}$$

$$=\frac{3R^3}{4r^3}-\frac{3R}{2r}=\frac{3R^3-6Rr^2}{4r^3}\overset{?}{\le}9.\left(\frac{9}{8}.\left(\frac{R}{r}\right)^3-8\right)^2-6=\frac{9(9R^3-64r^3)^2-384r^6}{64r^6}$$

$$\Leftrightarrow 3(9R^3-64r^3)^2-128r^6\overset{?}{\ge}16r^3(3R^3-6Rr^2)$$

$$\Leftrightarrow 243t^6-3472t^3+32t+12160\overset{?}{\ge}0$$

$$\Leftrightarrow (t-2)\left((t-2)(243t^4+972t^3+2916t^2+4304t+5552)+5024\right)\overset{?}{\ge}0\rightarrow\text{true}$$

$$\Rightarrow\left(\frac{m_a}{w_b}\right)^2+\left(\frac{w_b}{h_c}\right)^2+\left(\frac{h_c}{m_a}\right)^2\le 9.\left(\frac{9}{8}.\left(\frac{R}{r}\right)^3-8\right)^2-6$$

$\forall\,\Delta$ ABC,'' ='' iff Δ ABC is equilateral (QED)