ROMANIAN MATHEMATICAL MAGAZINE

In $\Delta ABC, n, m \in N;\ n \geq m+1$ then:

$$\frac{\tan^n \frac{A}{2}}{\sin^m A}+\frac{\tan^n \frac{B}{2}}{\sin^m B}+\frac{\tan^n \frac{C}{2}}{\sin^m C} \geq \frac{2^m}{3^{\frac{n+m-2}{2}}}$$

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Solution by Tapas Das-India

$$\sum \tan\frac{A}{2} \overset{Jensen}{\geq} 3\tan\frac{\pi}{6}=\sqrt{3}\ (1)\ and\ \sum \sin A=\frac{s}{R} \overset{Mitrinovic}{\leq} \frac{3\sqrt{3}}{2}\ (2)$$

- $for\ n=m+1\ we\ need\ to\ show\ \frac{\tan^{m+1} \frac{A}{2}}{\sin^m A}+\frac{\tan^{m+1} \frac{B}{2}}{\sin^m B}+\frac{\tan^{m+1} \frac{C}{2}}{\sin^m C} \geq \frac{2^m}{3^{\frac{2m-1}{2}}}$

$$\frac{\tan^{m+1} \frac{A}{2}}{\sin^m A}+\frac{\tan^{m+1} \frac{B}{2}}{\sin^m B}+\frac{\tan^{m+1} \frac{C}{2}}{\sin^m C} \overset{Radon}{\geq} \frac{\left(\sum \tan\frac{A}{2}\right)^{m+1}}{(\sum \sin A)^m} \overset{(1)\&(2)}{\geq} \frac{\left(\sqrt{3}\right)^{m+1}}{\left(\frac{3\sqrt{3}}{2}\right)^m}=\frac{2^m}{3^{\frac{2m-1}{2}}}$$

- $for\ n>m+1\ we\ need\ to\ show\ \frac{\tan^n \frac{A}{2}}{\sin^m A}+\frac{\tan^n \frac{B}{2}}{\sin^m B}+\frac{\tan^n \frac{C}{2}}{\sin^m C} \geq \frac{2^m}{3^{\frac{n+m-2}{2}}}$

$$\left(\frac{\tan^n \frac{A}{2}}{\sin^m A}+\frac{\tan^n \frac{B}{2}}{\sin^m B}+\frac{\tan^n \frac{C}{2}}{\sin^m C}\right)\left(\sum \sin A\right)^m (1+1+1)^{n-m-1} \overset{Holder}{\geq} \left(\sum \tan\frac{A}{2}\right)^n$$

$$\left(\frac{\tan^n \frac{A}{2}}{\sin^m A}+\frac{\tan^n \frac{B}{2}}{\sin^m B}+\frac{\tan^n \frac{C}{2}}{\sin^m C}\right) \geq \left(\sum \tan\frac{A}{2}\right)^n \frac{1}{(\sum \sin A)^m 3^{n-m-1}} \overset{(1)\&(2)}{\geq}$$

$$\geq \frac{\left(\sqrt{3}\right)^n}{\left(\frac{3\sqrt{3}}{2}\right)^m 3^{n-m-1}}=\frac{2^m}{3^{\frac{n+m-2}{2}}}$$