ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{n_a g_a}{m_a}+\frac{n_b g_b}{m_b}+\frac{n_c g_c}{m_c}\ge\frac{18r^2}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$m_a^2=s(s-a)+\frac{1}{4}(b-c)^2 \text{ and } n_a^2=s(s-a)+\frac{s(b-c)^2}{a}$$

$$\left[\begin{matrix}Reference: Bogdan\ Fustei\ Mohamed\ Amine\ Ben\ Ajiba\\ ABOUT\ NAGEL'S\ \ GERGONNE'SCEVIANS\end{matrix}\right]$$

$$n_a^2-m_a^2=(b-c)^2\left(\frac{s}{a}-\frac{1}{4}\right)=\frac{(b-c)^2(4s-a)}{4a}\ge 0$$

$$so\ \ n_a\ge m_a\ or\frac{n_a}{m_a}\ge 1\quad (1)$$

$$\frac{n_a g_a}{m_a}+\frac{n_b g_b}{m_b}+\frac{n_c g_c}{m_c}\overset{(1)}{\ge}.g_a+g_b+g_c\ge h_a+h_b+h_c\overset{AM-HM}{\ge}$$

$$\ge\frac{9}{\sum\frac{1}{h_a}}=\frac{9}{\frac{1}{r}}=9r=\frac{9r^2}{r}\overset{Euler}{\ge}\frac{18r^2}{R}$$

$$Equality\ holds\ for\ a=b=c$$