ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{\cot^7\frac{A}{2}}{\sin^3 A}+,\frac{\cot^7\frac{B}{2}}{\sin^3 B}+,\frac{\cot^7\frac{C}{2}}{\sin^3 C}\geq 216$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das

$$WLOG\ \ a\geq b\geq c\ then\ \sin A\geq \sin B\geq \sin C\ \ and\ \cot\frac{A}{2}\leq \cot\frac{B}{2}\leq \cot\frac{C}{2}$$

$$\frac{\cot^7\frac{A}{2}}{\sin^3 A}+,\frac{\cot^7\frac{B}{2}}{\sin^3 B}+,\frac{\cot^7\frac{C}{2}}{\sin^3 C}\overset{Chebyshev}{\geq}\frac{1}{3}\left(\sum\cot^7\frac{A}{2}\right)\left(\frac{1}{\sin^3 A}\right)=$$

$$=\frac{1}{3}\left(\sum\cot^7\frac{A}{2}\right)\left(\frac{1^4}{\sin^3 A}\right)\overset{CBS\ \&\ Radon}{\geq}$$

$$\geq\frac{1}{3}\cdot\frac{1}{3^6}\left(\sum\cot\frac{A}{2}\right)^7\frac{(1+1+1)^4}{(\sum\sin A)^3}=\frac{1}{27}\left(\frac{s}{r}\right)^7\left(\frac{R}{s}\right)^3=$$

$$=\frac{1}{27}\frac{s^4R^3}{r^7}\overset{Euler\ \&\ Mitrinovic}{\geq}\frac{1}{27}\frac{(27r^2)^2(2r)^3}{r^7}=216$$

$$Equality\ holds\ for\ A=B=C$$