ROMANIAN MATHEMATICAL MAGAZINE

If in $\Delta ABC, n, m \in N, n \geq m+1$ then:

$$\frac{\cot^n \frac{A}{2}}{\sin^m A}+,\frac{\cot^n \frac{B}{2}}{\sin^m B}+,\frac{\cot^n \frac{C}{2}}{\sin^m C} \geq 2^m 3^{\frac{n-m+2}{2}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$WLOG\ \ a \geq b \geq c\ then\ \sin A \geq \sin B \geq \sin C\ \ and\ \cot\frac{A}{2} \leq \cot\frac{B}{2} \leq \cot\frac{C}{2}$$

$$For\ n = m+1, we\ need\ to\ show \frac{\cot^{m+1} \frac{A}{2}}{\sin^m A}+,\frac{\cot^{m+1} \frac{B}{2}}{\sin^m B}+,\frac{\cot^{m+1} \frac{C}{2}}{\sin^m C} \geq 2^m 3^{\frac{3}{2}}$$

$Proof$:

$$\frac{\cot^{m+1} \frac{A}{2}}{\sin^m A}+,\frac{\cot^{m+1} \frac{B}{2}}{\sin^m B}+,\frac{\cot^{m+1} \frac{C}{2}}{\sin^m C} \overset{Radon}{\geq} \frac{\left(\sum \cot\frac{A}{2}\right)^{m+1}}{\left(\sum \sin A\right)^m} =$$

$$= \frac{\left(\frac{s}{r}\right)^{m+1}}{\left(\frac{s}{R}\right)^m} = s\left(\frac{R}{r}\right)^m \frac{1}{r} \overset{Mitrinovic\ \&\ Euler}{\geq} 2^m\, 3\sqrt{3}r\ \ \frac{1}{r} = 2^m 3^{\frac{3}{2}}$$

$For\ n > m+1$

$$\frac{\cot^n \frac{A}{2}}{\sin^m A}+,\frac{\cot^n \frac{B}{2}}{\sin^m B}+,\frac{\cot^n \frac{C}{2}}{\sin^m C} \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum \cot^n \frac{A}{2}\right)\left(\frac{1}{\sin^m A}\right) =$$

$$= \frac{1}{3}\left(\sum \cot^n \frac{A}{2}\right)\left(\frac{1^{m+1}}{\sin^m A}\right) \overset{CBS\ \&\ Radon}{\geq} \frac{1}{3} \cdot \frac{1}{3^{n-1}}\left(\sum \cot\frac{A}{2}\right)^n \frac{(1+1+1)^{m+1}}{\left(\sum \sin A\right)^m} =$$

$$= \frac{1}{3^n}\left(\frac{s}{r}\right)^n \left(\frac{R}{s}\right)^m 3^{m+1} = \frac{1}{3^n}\frac{s^{n-m}R^m}{r^n} 3^{m+1} \overset{Euler\ \&\ Mitrinovic}{\geq}$$

$$\geq \frac{1}{3^n}\frac{\left(3\sqrt{3}r\right)^{n-m}(2r)^m}{r^n} 3^{m+1} = 2^m \frac{(3)^{\frac{3n-3m}{2}}}{3^n} 3^{m+1} = 2^m 3^{\frac{n-m+2}{2}}$$

$$Equality\ holds\ for\ A = B = C$$