ROMANIAN MATHEMATICAL MAGAZINE

If in $\Delta ABC\ n,m\in N, n\geq m+1$ then:

$$\frac{n_a^n}{m_a^m}+\frac{n_b^n}{m_b^m}+\frac{n_c^n}{m_c^m}\geq 2^m3^{n-m+1}\frac{r^n}{R^m}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$m_a^2=s(s-a)+\frac{1}{4}(b-c)^2 \text{ and } n_a^2=s(s-a)+\frac{s(b-c)^2}{a}$$

$$\left[\begin{matrix}Reference: Bogdan\ Fustei, Mohamed\ Amine\ Ben\ Ajiba\\ ABOUT\ NAGEL'S\ \ AND\ GERGONNE'S\ CEVIANS\end{matrix}\right]$$

$$n_a^2-m_a^2=(b-c)^2\left(\frac{s}{a}-\frac{1}{4}\right)=\frac{(b-c)^2(4s-a)}{4a}\geq 0\text{ , so } n_a\geq m_a \text{ or } \frac{n_a}{m_a}\geq 1\quad (1)$$

$$h_a+h_b+h_c\overset{AM-HM}{\geq}\frac{9}{\sum\frac{1}{h_a}}=\frac{9}{\frac{1}{r}}=9r\ \ (2)$$

Case 1: $n=m+1$

$$We\ need\ to\ show\ \frac{n_a^{m+1}}{m_a^m}+\frac{n_b^{m+1}}{m_b^m}+\frac{n_c^{m+1}}{m_c^m}\geq 2^m3^2\frac{r^{m+1}}{R^m}$$

$$\frac{n_a^{m+1}}{m_a^m}+\frac{n_b^{m+1}}{m_b^m}+\frac{n_c^{m+1}}{m_c^m}=\sum\left(\left(\frac{n_a}{m_a}\right)^m n_a\right)\overset{(1)}{\geq}\sum n_a\geq\sum h_a\overset{(2)}{\geq}9r=$$

$$=3^2r\frac{r^m}{r^m}\overset{Euler}{\geq}3^2\frac{r^{m+1}}{\left(\frac{R}{2}\right)^m}=2^m3^2\frac{r^{m+1}}{R^m}$$

Case2: $n>m+1$

$$We\ need\ to\ show\ \frac{n_a^n}{m_a^m}+\frac{n_b^n}{m_b^m}+\frac{n_c^n}{m_c^m}\geq 2^m3^{n-m+1}\frac{r^n}{R^m}$$

$$\frac{n_a^n}{m_a^m}+\frac{n_b^n}{m_b^m}+\frac{n_c^n}{m_c^m}=\sum\frac{n_a^n}{m_a^m}=\sum\left(\left(\frac{n_a}{m_a}\right)^m(n_a)^{n-m}\right)\geq$$

$$\geq\sum n_a^{n-m}\overset{CBS}{\geq}\frac{1}{3^{n-m-1}}(n_a+n_b+n_c)^{n-m}\geq$$

$$\geq\frac{1}{3^{n-m-1}}(h_a+h_b+h_c)^{n-m}\overset{(1)}{\geq}\frac{1}{3^{n-m-1}}(9r)^{n-m}=$$

$$=3^{n-m+1}.r^{n-m}.\frac{r^m}{r^m}\overset{Euler}{\geq}3^{n-m+1}\frac{r^n}{R^m}2^m=2^m3^{n-m+1}\frac{r^n}{R^m}$$

Equality holds for $a=b=c$