ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{48r^4}{3R^4-32r^4} \le \left(\frac{m_a w_b}{w_c h_a}\right)^2+\left(\frac{w_b h_c}{h_a m_b}\right)^2+\left(\frac{h_c m_a}{m_b w_c}\right)^2 \le 9.\left(\frac{9}{8}.\left(\frac{R}{r}\right)^3-8\right)^2-6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{am_a}\sum_{cyc}a^2 \ge 2\sqrt{3} \Leftrightarrow \frac{1}{a^2m_a^2} \ge \frac{12}{\left(\sum_{cyc}a^2\right)^2}$$

$$\Leftrightarrow \left(\sum_{cyc}a^2\right)^2 - 3a^2(2b^2+2c^2-a^2) \ge 0 \Leftrightarrow \left(\sum_{cyc}a^2\right)^2 - 3a^2\left(2\sum_{cyc}a^2-3a^2\right)$$

$$\ge 0 \Leftrightarrow \left(\sum_{cyc}a^2\right)^2 - 6a^2\sum_{cyc}a^2+9a^4 \ge 0 \Leftrightarrow \left(\sum_{cyc}a^2-3a^2\right)^2 \ge 0$$

$$\Leftrightarrow (b^2+c^2-2a^2)^2 \ge 0 \to \text{true} \Rightarrow am_a \le \frac{\sum_{cyc}a^2}{2\sqrt{3}} \text{ and analogs} \to (1)$$

$$\text{Also, } 2s^2 \overset{\text{Gerretsen}}{\ge} 27Rr+5r(R-2r) \overset{\text{Euler}}{\ge} 27Rr \therefore 2s^2 \ge 27Rr \to (2)$$

$$\text{Now, } \left(\frac{m_a w_b}{w_c h_a}\right)^2+\left(\frac{w_b h_c}{h_a m_b}\right)^2+\left(\frac{h_c m_a}{m_b w_c}\right)^2$$

$$\le \left(\frac{m_a}{h_a}\right)^2\left(\frac{m_b}{h_c}\right)^2+\left(\frac{m_b}{h_b}\right)^2\left(\frac{m_c}{h_a}\right)^2+\left(\frac{m_c}{h_c}\right)^2\left(\frac{m_a}{h_b}\right)^2$$

$$= \frac{(am_a)^2}{4r^2s^2}.\frac{c^2(2c^2+2a^2+2b^2-3b^2)}{16r^2s^2}+\frac{(bm_b)^2}{4r^2s^2}.\frac{a^2(2a^2+2b^2+2c^2-3c^2)}{16r^2s^2}$$

$$+\frac{(cm_c)^2}{4r^2s^2}.\frac{b^2(2b^2+2c^2+2a^2-3a^2)}{16r^2s^2} \overset{\text{via }(1)}{\le} \frac{\left(\sum_{cyc}a^2\right)^2}{48r^2s^2.16r^2s^2}.\left(2\left(\sum_{cyc}a^2\right)^2 - 3\sum_{cyc}a^2b^2\right)$$

$$\overset{\substack{\text{Leibnitz}\\ \text{and}\\ (2)}}{\le} \frac{81R^4\left(162R^4-3abc\left(\sum_{cyc}a\right)\right)}{384r^4s^2.27Rr} = \frac{R^3.162R^4}{128r^5s^2}-\frac{R^3.24Rrs^2}{128r^5s^2}$$

$$\overset{\text{via }(2)}{\le} \frac{R^3.162R^4}{64r^5.27Rr}-\frac{6R^4}{32r^4} = \frac{R^3(3R^3-6Rr^2)}{32r^6} \overset{?}{\le} 9.\left(\frac{9}{8}.\left(\frac{R}{r}\right)^3-8\right)^2-6$$

$$= \frac{9(9R^3-64r^3)^2-384r^6}{64r^6} \Leftrightarrow 241t^6+4t^4-3456t^3+12160 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(241t^4+964t^3+2896t^2+4272t+5504)+4928\right) \overset{?}{\ge} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \therefore \left(\frac{m_a w_b}{w_c h_a}\right)^2 + \left(\frac{w_b h_c}{h_a m_b}\right)^2 + \left(\frac{h_c m_a}{m_b w_c}\right)^2 \leq 9.\left(\frac{9}{8}.\left(\frac{R}{r}\right)^3 - 8\right)^2 - 6$$

$$\text{Again}, \left(\frac{m_a w_b}{w_c h_a}\right)^2 + \left(\frac{w_b h_c}{h_a m_b}\right)^2 + \left(\frac{h_c m_a}{m_b w_c}\right)^2 \overset{A-G}{\geq} 3.\sqrt[3]{\frac{m_a^4 w_b^4 h_c^4}{w_c^4 h_a^4 m_b^4}} \geq 3.\sqrt[3]{\frac{h_a^4 h_b^4 h_c^4}{m_c^4 m_a^4 m_b^4}}$$

$$\overset{m_a m_b m_c \leq \frac{Rs^2}{2}}{\geq} 3.\sqrt[3]{\frac{\left(\frac{2r^2s^2}{R}\right)^4}{\left(\frac{Rs^2}{2}\right)^4}} = 3.\frac{4r^2}{R^2}.\sqrt[3]{\frac{4r^2}{R^2}} \overset{?}{\geq} \frac{48r^4}{3R^4 - 32r^4} \Leftrightarrow \frac{4r^2}{R^2} \overset{?}{\geq} \frac{64r^6R^6}{(3R^4 - 32r^4)^3}$$

$$\Leftrightarrow 27t^{12} - 880t^8 + 9216t^4 - 32768 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)^2 \begin{pmatrix} 27t^{10} + 108t^9 + 324t^8 + 864t^7 + 1280t^6 \\ +1664t^5 + 1536t^4 - 512t^3 + 1024t^2 + 6144t + 20480 \end{pmatrix}$$

$$+57344(t-2) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\therefore \left(\frac{m_a w_b}{w_c h_a}\right)^2 + \left(\frac{w_b h_c}{h_a m_b}\right)^2 + \left(\frac{h_c m_a}{m_b w_c}\right)^2 \geq \frac{48r^4}{3R^4 - 32r^4}$$

$$\Rightarrow \frac{48r^4}{3R^4 - 32r^4} \leq \left(\frac{m_a w_b}{w_c h_a}\right)^2 + \left(\frac{w_b h_c}{h_a m_b}\right)^2 + \left(\frac{h_c m_a}{m_b w_c}\right)^2 \leq 9.\left(\frac{9}{8}.\left(\frac{R}{r}\right)^3 - 8\right)^2 - 6$$

$\forall \Delta ABC, '' ='' \text{ iff } \Delta ABC$ is equilateral (QED)

Proof of $m_a m_b m_c \leq \frac{Rs^2}{2}$

$$m_a^2 m_b^2 m_c^2 = \frac{1}{64}(2b^2 + 2c^2 - a^2)(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2)$$

$$\overset{(1)}{=} \frac{1}{64}\left\{-4\sum_{cyc} a^6 + 6\left(\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right) + 3a^2b^2c^2\right\}$$

$$\text{Now}, \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= \left(\sum_{cyc} a^2\right)^3 - 3\left(2a^2b^2c^2 + \sum_{cyc}\left(a^2b^2\left(\sum a^2 - c^2\right)\right)\right)$$

$$= \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\therefore \sum_{cyc} a^6 \overset{(2)}{=} \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right) \overset{(3)}{=}$$

$$\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \therefore (1),(2),(3) \Rightarrow m_a^2m_b^2m_c^2$$

$$= \frac{1}{64}\begin{pmatrix} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{pmatrix}$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\begin{Bmatrix} -32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2 \\ -576Rrs^2(s^2-4Rr-r^2) - 432R^2r^2s^2 \end{Bmatrix}$$

$$= \frac{1}{16}\{s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3\}$$

$$\le \frac{R^2s^4}{4} \Leftrightarrow$$

$$s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3 \overset{(\bullet)}{\le} 0$$

Now, LHS of $(\bullet) \overset{\text{Gerretsen}}{\le} -s^4(8Rr - 36r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4)$

$-r^3(4R+r)^3 \overset{?}{\le} 0$

$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3 \underset{(\bullet\bullet)}{\overset{?}{\ge}} 20rs^4$

Now, LHS of $(\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\ge}}_{(a)} s^2(16Rr - 5r^2)(8R - 16r)$

$+s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3$ *and*

RHS of $(\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\le}}_{(b)} 20rs^2(4R^2 + 4Rr + 3r^2)$

$(a),(b) \Rightarrow$ **in order to prove** $(\bullet\bullet)$, **it suffices to prove :**

$s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3$

$\ge 20rs^2(4R^2 + 4Rr + 3r^2)$

$\Leftrightarrow s^2(108R^2 - 256Rr + 53r^2) + r(4R+r)^3 \ge 0$

$$\Leftrightarrow s^2\left(108R^2-256Rr+80r^2\right)+r(4R+r)^3\overset{(\bullet\bullet\bullet)}{\geq}27r^2s^2$$

Now, LHS of $(\bullet\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\geq}}_{(c)}\left(108R^2-256Rr+80r^2\right)\left(16Rr-5r^2\right)+r(4R+r)^3$

and RHS of $(\bullet\bullet\bullet)\underbrace{\overset{\text{Geretsen}}{\leq}}_{(d)}27r^2\left(4R^2+4Rr+3r^2\right)$

$(c),(d)\Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$$\left(108R^2-256Rr+80r^2\right)\left(16Rr-5r^2\right)+r(4R+r)^3\geq 27r^2\left(4R^2+4Rr+3r^2\right)$$

$$\Leftrightarrow 224t^3-587t^2+308t-60\geq 0\ \left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(224t+309)+648\right)\geq 0\rightarrow \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)$$

$$\Rightarrow(\bullet)\text{ is true}\Rightarrow m_a^2m_b^2m_c^2\leq\frac{R^2s^4}{4}\Rightarrow m_am_bm_c\leq\frac{Rs^2}{2}\ (\text{QED})$$