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In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{\sin\frac{C}{2}.\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+\sin\frac{B}{2}.\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{B}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)}\geq 6$$

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$\forall$ A, B, C > 0, $(A+B), (B+C), (C+A)$ form sides of a triangle
$(\because (A+B)+(B+C) > (C+A)$ and analogs$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$= 6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \rightarrow (1)$$

$$\text{Now, } \forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x \rightarrow \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow (2)$$

We have :

$$\frac{\sin\frac{C}{2}.\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+\sin\frac{B}{2}.\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{B}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)}+$$

$$\frac{\sin\frac{C}{2}.\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}+\sin\frac{A}{2}.\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{C}{2}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}+$$

$$\frac{\sin\frac{B}{2}.\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}+\sin\frac{A}{2}.\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{A}{2}\left(\sin\frac{C}{2}+\sin\frac{A}{2}\right)}$$

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$$=\frac{\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{B}{2}}+\frac{\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{C}{2}}}{\frac{\sin\frac{A}{2}+\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\frac{\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin\frac{A}{2}}+\frac{\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{C}{2}}}{\frac{\sin\frac{B}{2}+\sin\frac{C}{2}}{\sin\frac{A}{2}}}+\frac{\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin\frac{A}{2}}+\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{B}{2}}}{\frac{\sin\frac{C}{2}+\sin\frac{A}{2}}{\sin\frac{B}{2}}}$$

$$=\frac{\sin\frac{C}{2}}{\sin\frac{A}{2}+\sin\frac{B}{2}}.\left(\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{B}{2}}+\frac{\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{C}{2}}\right)+$$

$$\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}.\left(\frac{\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{C}{2}}+\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin\frac{A}{2}}\right)+$$

$$\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}+\sin\frac{A}{2}}.\left(\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin\frac{A}{2}}+\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{B}{2}}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(\begin{array}{c} x=\sin\frac{C}{2}, y=\sin\frac{A}{2}, z=\sin\frac{B}{2}, \\ A=\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin\frac{A}{2}}, B=\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{B}{2}}, C=\frac{\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{C}{2}}\end{array}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{via\ (1)\ and\ (2)}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}$$

$$=\sqrt{3\sum_{cyc}\left(\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin\frac{A}{2}}.\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{B}{2}}\right)}\overset{A-G}{\geq}$$

$$3.\sqrt{\sqrt[3]{\frac{\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\left(\sin\frac{C}{2}+\sin\frac{A}{2}\right)}{\left(\prod_{cyc}\sin\frac{A}{2}\right).\left(\prod_{cyc}\sin\frac{A}{2}\right)}}}\overset{\text{Cesaro}}{\geq}3.\sqrt[6]{\frac{8.4R}{r}}\overset{\text{Euler}}{\geq}3.\sqrt[6]{64}=6$$

$$\therefore\frac{\sin\frac{C}{2}.\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+\sin\frac{B}{2}.\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{B}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)}+$$

$$\frac{\sin\frac{C}{2}.\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}+\sin\frac{A}{2}.\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{C}{2}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}+$$

$$\frac{\sin\frac{B}{2}.\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}+\sin\frac{A}{2}.\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin\frac{A}{2}\left(\sin\frac{C}{2}+\sin\frac{A}{2}\right)}\geq 6$$

$\forall\ \Delta\ ABC,''=''$ **iff** Δ **ABC is equilateral (QED)**