ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{b^3\cot\frac{C}{2}+c^3\cot\frac{B}{2}}{a(b^2+c^2)}+\frac{c^3\cot\frac{A}{2}+a^3\cot\frac{C}{2}}{b(c^2+a^2)}+\frac{a^3\cot\frac{B}{2}+b^3\cot\frac{A}{2}}{c(a^2+b^2)}\geq 3\sqrt{3}$$

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$\forall$ A, B, C >0, $(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

Now, $\forall x,y,z>0$, $\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$

Via Bergstrom, LHS of $(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

We have : $\frac{b^3\cot\frac{C}{2}+c^3\cot\frac{B}{2}}{a(b^2+c^2)}+\frac{c^3\cot\frac{A}{2}+a^3\cot\frac{C}{2}}{b(c^2+a^2)}+\frac{a^3\cot\frac{B}{2}+b^3\cot\frac{A}{2}}{c(a^2+b^2)}$

$$=\frac{\frac{ab\cot\frac{C}{2}}{c^2}+\frac{ac\cot\frac{B}{2}}{b^2}}{a^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)}+\frac{\frac{bc\cot\frac{A}{2}}{a^2}+\frac{ab\cot\frac{C}{2}}{c^2}}{b^2\left(\frac{1}{c^2}+\frac{1}{a^2}\right)}+\frac{\frac{ac\cot\frac{B}{2}}{b^2}+\frac{bc\cot\frac{A}{2}}{a^2}}{c^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$$

$$=\frac{\frac{1}{a^2}}{\frac{1}{b^2}+\frac{1}{c^2}}\cdot\left(\frac{ca\cot\frac{B}{2}}{b^2}+\frac{ab\cot\frac{C}{2}}{c^2}\right)+\frac{\frac{1}{b^2}}{\frac{1}{c^2}+\frac{1}{a^2}}\cdot\left(\frac{ab\cot\frac{C}{2}}{c^2}+\frac{bc\cot\frac{A}{2}}{a^2}\right)$$

$$+\frac{\frac{1}{c^2}}{\frac{1}{a^2}+\frac{1}{b^2}}\cdot\left(\frac{bc\cot\frac{A}{2}}{a^2}+\frac{ca\cot\frac{B}{2}}{b^2}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{1}{a^2},y=\frac{1}{b^2},z=\frac{1}{c^2},A=\frac{bc\cot\frac{A}{2}}{a^2},B=\frac{ca\cot\frac{B}{2}}{b^2},C=\frac{ab\cot\frac{C}{2}}{c^2}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\left(\frac{bc\cot\frac{A}{2}}{a^2}.\frac{ca\cot\frac{B}{2}}{b^2}\right)}$$

$$=\sqrt{3\sum_{cyc}\left(\frac{c^2}{ab}.\cot\frac{A}{2}\cot\frac{B}{2}\right)}\overset{A-G}{\geq}3.\sqrt[6]{\frac{c^2}{ab}.\frac{a^2}{bc}.\frac{b^2}{ca}.\left(\prod_{cyc}\cot\frac{A}{2}\right)^2}=3.\sqrt[6]{\frac{s^6}{(r_ar_br_c)^2}}$$

$$=3.\sqrt[6]{\frac{s^2}{r^2}}\overset{\text{Mitrinovic}}{\geq}3.\sqrt[6]{27}=3\sqrt{3}$$

$$\therefore\frac{b^3\cot\frac{C}{2}+c^3\cot\frac{B}{2}}{a(b^2+c^2)}+\frac{c^3\cot\frac{A}{2}+a^3\cot\frac{C}{2}}{b(c^2+a^2)}+\frac{a^3\cot\frac{B}{2}+b^3\cot\frac{A}{2}}{c(a^2+b^2)}\geq3\sqrt{3}$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)