ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{\left(\sin\frac{A}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}\cdot\frac{b^2\sin\frac{B}{2}+c^2\sin\frac{C}{2}}{c\sin\frac{B}{2}+b\sin\frac{C}{2}}+\frac{\left(\sin\frac{B}{2}\right)}{\sin\frac{A}{2}\sin\frac{C}{2}}\cdot\frac{a^2\sin\frac{A}{2}+c^2\sin\frac{C}{2}}{c\sin\frac{A}{2}+a\sin\frac{C}{2}}+\frac{\left(\sin\frac{C}{2}\right)}{\sin\frac{B}{2}\sin\frac{A}{2}}\cdot\frac{a^2\sin\frac{A}{2}+b^2\sin\frac{B}{2}}{b\sin\frac{A}{2}+a\sin\frac{B}{2}}\ge 12\sqrt{3}r$$

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Solution by Tapas Das-India

$$WLOG\ a\ge b\ge c\text{, then. }\sin\frac{A}{2}\ge\sin\frac{B}{2}\ge\sin\frac{C}{2}\ and\ a\sin\frac{A}{2}\ge b\sin\frac{B}{2}\ge c\sin\frac{C}{2}$$

$$\frac{b^2\sin\frac{B}{2}+c^2\sin\frac{C}{2}}{c\sin\frac{B}{2}+b\sin\frac{C}{2}}=\frac{b(b\sin\frac{B}{2})+c(c\sin\frac{C}{2})}{c\sin\frac{B}{2}+b\sin\frac{C}{2}}\overset{Chebyshev}{\ge}$$

$$\ge\frac{\frac{1}{2}(b+c)\left(b\sin\frac{B}{2}+c\sin\frac{C}{2}\right)}{\left(b\sin\frac{B}{2}+c\sin\frac{C}{2}\right)}=\frac{1}{2}(b+c)\overset{AM-GM}{\ge}\frac{2\sqrt{ab}}{2}=\sqrt{ab}$$

$$Similarly:\ \frac{a^2\sin\frac{A}{2}+c^2\sin\frac{C}{2}}{c\sin\frac{A}{2}+a\sin\frac{C}{2}}\ge\sqrt{ac}\ and\ \frac{a^2\sin\frac{A}{2}+b^2\sin\frac{B}{2}}{b\sin\frac{A}{2}+a\sin\frac{B}{2}}\ge\sqrt{ab}$$

using above relation:

$$\frac{\left(\sin\frac{A}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}\cdot\frac{b^2\sin\frac{B}{2}+c^2\sin\frac{C}{2}}{c\sin\frac{B}{2}+b\sin\frac{C}{2}}+\frac{\left(\sin\frac{B}{2}\right)}{\sin\frac{A}{2}\sin\frac{C}{2}}\cdot\frac{a^2\sin\frac{A}{2}+c^2\sin\frac{C}{2}}{c\sin\frac{A}{2}+a\sin\frac{C}{2}}+\frac{\left(\sin\frac{C}{2}\right)}{\sin\frac{B}{2}\sin\frac{A}{2}}\cdot\frac{a^2\sin\frac{A}{2}+b^2\sin\frac{B}{2}}{b\sin\frac{A}{2}+a\sin\frac{B}{2}}\ge$$

$$\ge\frac{\left(\sin\frac{A}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}.\sqrt{bc}+\frac{\left(\sin\frac{B}{2}\right)}{\sin\frac{A}{2}\sin\frac{C}{2}}.\sqrt{ac}+\frac{\left(\sin\frac{C}{2}\right)}{\sin\frac{B}{2}\sin\frac{A}{2}}.\sqrt{ab}\overset{AM-GM}{\ge}$$

$$\ge 3\sqrt[3]{\frac{\left(\sin\frac{A}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}.\sqrt{bc}.\frac{\left(\sin\frac{B}{2}\right)}{\sin\frac{A}{2}\sin\frac{C}{2}}.\sqrt{ac}.\frac{\left(\sin\frac{C}{2}\right)}{\sin\frac{B}{2}\sin\frac{A}{2}}.\sqrt{ab}}=3\sqrt[3]{\frac{abc}{\prod\sin\frac{A}{2}}}$$

$$=3\left(4Rrs\cdot\frac{4R}{r}\right)^{\frac{1}{3}}\overset{Euler\ \&\ Mitrinovic}{\ge}3(4\cdot 2r\cdot r.3\sqrt{3}r\cdot 4\cdot 2r)^{\frac{1}{3}}=$$

$$=3\left(64\,r^3(\sqrt{3}\,)^3\right)^{\frac{1}{3}}=3.4.\sqrt{3}r=12\sqrt{3}r$$

Equality for $a=b=c$