ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC and $\forall$ n $\in \mathbb{N}$, the $following$ $relationship$ $holds$:

$$\sum_{cyc}\frac{b^{2n+1}\cot\frac{C}{2}+c^{2n+1}\cot\frac{B}{2}}{a(b^{n+1}+c^{n+1})}\geq 3\sqrt{3}(abc)^{\frac{n-1}{3}}$$

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$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle $(\because$ (A + B) + (B + C) > (C + A) and $analogs)$ $\Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with $area$ F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have : }\frac{b^{2n+1}\cot\frac{C}{2}+c^{2n+1}\cot\frac{B}{2}}{a(b^{n+1}+c^{n+1})}+\frac{c^{2n+1}\cot\frac{A}{2}+a^{2n+1}\cot\frac{C}{2}}{b(c^{n+1}+a^{n+1})}+\frac{a^{2n+1}\cot\frac{B}{2}+b^{2n+1}\cot\frac{A}{2}}{c(a^{n+1}+b^{n+1})}$$

$$=\frac{\frac{b^n}{c^{n+1}}\cot\frac{C}{2}+\frac{c^n}{b^{n+1}}\cot\frac{B}{2}}{a\left(\frac{1}{c^{n+1}}+\frac{1}{b^{n+1}}\right)}+\frac{\frac{c^n}{a^{n+1}}\cot\frac{A}{2}+\frac{a^n}{c^{n+1}}\cot\frac{C}{2}}{b\left(\frac{1}{a^{n+1}}+\frac{1}{c^{n+1}}\right)}+\frac{\frac{a^n}{b^{n+1}}\cot\frac{B}{2}+\frac{b^n}{a^{n+1}}\cot\frac{A}{2}}{c\left(\frac{1}{b^{n+1}}+\frac{1}{a^{n+1}}\right)}$$

$$=\frac{\frac{1}{a^{n+1}}}{\frac{1}{b^{n+1}}+\frac{1}{c^{n+1}}}\cdot\left(\frac{c^na^n\cot\frac{B}{2}}{b^{n+1}}+\frac{a^nb^n\cot\frac{C}{2}}{c^{n+1}}\right)+\frac{\frac{1}{b^{n+1}}}{\frac{1}{c^{n+1}}+\frac{1}{a^{n+1}}}\cdot\left(\frac{a^nb^n\cot\frac{C}{2}}{c^{n+1}}+\frac{b^nc^n\cot\frac{A}{2}}{a^{n+1}}\right)$$

$$+\frac{\frac{1}{c^{n+1}}}{\frac{1}{a^{n+1}}+\frac{1}{b^{n+1}}}\cdot\left(\frac{b^nc^n\cot\frac{A}{2}}{a^{n+1}}+\frac{c^na^n\cot\frac{B}{2}}{b^{n+1}}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{1}{a^{n+1}}, y=\frac{1}{b^{n+1}}, z=\frac{1}{c^{n+1}}, A=\frac{b^nc^n\cot\frac{A}{2}}{a^{n+1}}, B=\frac{c^na^n\cot\frac{B}{2}}{b^{n+1}}, C=\frac{a^nb^n\cot\frac{C}{2}}{c^{n+1}}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}$$

$$=\sqrt{3\sum_{cyc}\left(\frac{b^nc^n\cot\frac{A}{2}}{a^{n+1}}.\frac{c^na^n\cot\frac{B}{2}}{b^{n+1}}\right)}=\sqrt{\frac{3(abc)^n}{(abc)^{n+1}}.\sum_{cyc}\left(c^{2n+1}.\cot\frac{A}{2}\cot\frac{B}{2}\right)}$$

$$\overset{A-G}{\geq}3.\sqrt{\frac{1}{abc}.\sqrt[3]{(abc)^{2n+1}.\left(\prod_{cyc}\cot\frac{A}{2}\right)^2}}=3\sqrt{(abc)^{\frac{2n-2}{3}}.\sqrt[3]{\frac{s^6}{(r_ar_br_c)^2}}}$$

$$=3(abc)^{\frac{n-1}{3}}.\sqrt[6]{\frac{s^6}{r^2s^4}}=3(abc)^{\frac{n-1}{3}}.\sqrt[3]{\frac{s}{r}}\overset{\text{Mitrinovic}}{\geq}=3(abc)^{\frac{n-1}{3}}.\sqrt{3}$$

$$\therefore\frac{b^{2n+1}\cot\frac{C}{2}+c^{2n+1}\cot\frac{B}{2}}{a(b^{n+1}+c^{n+1})}+\frac{c^{2n+1}\cot\frac{A}{2}+a^{2n+1}\cot\frac{C}{2}}{b(c^{n+1}+a^{n+1})}+\frac{a^{2n+1}\cot\frac{B}{2}+b^{2n+1}\cot\frac{A}{2}}{c(a^{n+1}+b^{n+1})}$$

$$\geq 3\sqrt{3}(abc)^{\frac{n-1}{3}}\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$