ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{h_a+w_b}{m_a r_b}+\frac{h_b+w_c}{m_b r_c}+\frac{h_c+w_a}{m_c r_a}\geq\frac{8r}{R^2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a+w_b}{m_a r_b}+\frac{h_b+w_c}{m_b r_c}+\frac{h_c+w_a}{m_c r_a}\geq$$

$$\geq\frac{h_a}{m_a r_b}+\frac{h_b}{m_b r_c}+\frac{h_c}{m_c r_a}+\frac{h_b}{m_a r_b}+\frac{h_c}{m_b r_c}+\frac{h_a}{m_c r_a}\overset{A-G}{\geq}$$

$$\geq 6\sqrt[6]{\frac{\left(\prod_{cyc}h_a\right)^2}{\left(\prod_{cyc}m_a\right)^2\left(\prod_{cyc}r_a\right)^2}}\overset{m_a m_b m_c\leq\frac{Rs^2}{2}}{\geq}6.\sqrt[3]{\frac{2r^2s^2}{R.\frac{Rs^2}{2}.rs^2}}\overset{\text{Mitrinovic}}{\geq}$$

$$\geq 6.\sqrt[3]{\frac{2r^2s^2}{R.\frac{Rs^2}{2}.r.\frac{27R^2}{4}}}=2\sqrt[3]{\frac{16r}{R^4}}\overset{?}{\geq}\frac{8r}{R^2}$$

$$\Leftrightarrow\frac{16r}{R^4}\overset{?}{\geq}\frac{64r^3}{R^6}\Leftrightarrow R^2\overset{?}{\geq}4r^2\to\text{true via Euler}\therefore\frac{h_a+w_b}{m_a r_b}+\frac{h_b+w_c}{m_b r_c}+\frac{h_c+w_a}{m_c r_a}$$

$$\geq\frac{8r}{R^2}\ \forall\,\Delta\text{ ABC},''=''\text{ iff }\Delta\text{ ABC is equilateral}$$

$$\boxed{\text{Proof of } m_a m_b m_c\leq\frac{Rs^2}{2}}$$

$$m_a^2m_b^2m_c^2=\frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$=\frac{1}{64}\left(-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right)\to(1)$$

$$\text{Now,}\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3-3\left(2a^2b^2c^2+\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \to (2)$$

$$\text{Also, } \sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right) =$$

$$\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \to (3) \therefore (1),(2),(3) \Rightarrow m_a^2m_b^2m_c^2$$

$$= \frac{1}{64}\begin{pmatrix} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{pmatrix}$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\begin{pmatrix} -32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2 \\ -576Rrs^2(s^2-4Rr-r^2) - 432R^2r^2s^2 \end{pmatrix}$$

$$= \frac{1}{16}\left(s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3\right)$$

$$\leq \frac{R^2s^4}{4} \Leftrightarrow$$

$$s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3 \overset{(\bullet)}{\leq} 0$$

$$\text{Now, LHS of } (\bullet) \overset{\text{Gerretsen}}{\leq} -s^4(8Rr - 36r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4)$$

$$-r^3(4R+r)^3 \overset{?}{\leq} 0$$

$$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3 \underset{(\bullet\bullet)}{\overset{?}{\geq}} 20rs^4$$

$$\text{Now, LHS of } (\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(*)} s^2(16Rr - 5r^2)(8R - 16r)$$

$$+s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3 \text{ and}$$

$$\text{RHS of } (\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\leq}}_{(**)} 20rs^2(4R^2+4Rr+3r^2)$$

$(*), (**) \Rightarrow$ **in order to prove** $(\bullet\bullet)$, **it suffices to prove :**

$$s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$$
$$\geq 20rs^2(4R^2+4Rr+3r^2)$$
$$\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3 \geq 0$$
$$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\geq} 27r^2s^2$$

$$\text{Now, LHS of } (\bullet\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(***)} (108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$$

$$\text{and RHS of } (\bullet\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\leq}}_{(****)} 27r^2(4R^2+4Rr+3r^2)$$

$(***), (****) \Rightarrow$ **in order to prove** $(\bullet\bullet\bullet)$, **it suffices to prove :**

$$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3 \geq 27r^2(4R^2+4Rr+3r^2)$$
$$\Leftrightarrow 224t^3-587t^2+308t-60 \geq 0 \left(\text{where } t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\big((t-2)(224t+309)+648\big) \geq 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet)$$
$$\Rightarrow (\bullet) \text{ is true} \Rightarrow m_a^2m_b^2m_c^2 \leq \frac{R^2s^4}{4} \Rightarrow m_am_bm_c \leq \frac{Rs^2}{2} \text{ (QED)}$$