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In ΔABC the following relationship holds:

$$\sum \frac{\cot\frac{B}{2}\frac{\sin C}{\sin A}+\cot\frac{C}{2}\frac{\sin B}{\sin A}}{\sin C+\sin B}\ge 6$$

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Solution by Tapas Das-India

$$WLOG\ a\ge b\ge c\ then\ \cot\frac{A}{2}\le\cot\frac{B}{2}\le\cot\frac{C}{2}\ and\ \sin A\ge\sin B\ge\sin C$$

$$\sum \frac{\cot\frac{B}{2}\frac{\sin C}{\sin A}+\cot\frac{C}{2}\frac{\sin B}{\sin A}}{\sin C+\sin B}\overset{Chebyshev}{\ge}\sum\frac{\frac{1}{\sin A}\left(\frac{1}{2}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)(\sin C+\sin B)\right)}{\sin C+\sin B}=$$

$$=\frac{1}{2}\sum\frac{1}{\sin A}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)\overset{AM-GM}{\ge}\frac{1}{2}\sum\frac{1}{\sin A}2\sqrt{\left(\cot\frac{B}{2}\cot\frac{C}{2}\right)}=$$

$$=\sum\frac{1}{\sin A}\sqrt{\left(\cot\frac{B}{2}\cot\frac{C}{2}\right)}\overset{AM-GM}{\ge}3\sqrt[3]{\frac{\prod\cot\frac{A}{2}}{\prod\sin A}}=3\sqrt[3]{\frac{\frac{s}{r}}{\frac{sr}{2R^2}}}=3\sqrt[3]{2\left(\frac{R}{r}\right)^2}\overset{Euler}{\ge}3\sqrt[3]{8}=6$$

$$Equality\ holds\ for\ A=B=C$$