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In ΔABC the following relationship holds:

$$\sum \frac{\cot^n \frac{B \sin C}{2 \sin A} + \cot^n \frac{C \sin B}{2 \sin A}}{\sin C + \sin B} \geq 6 \cdot 3^{\frac{n-1}{2}}, n \in \mathbb{N}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

WLOG $a \geq b \geq c$ then $\cot \frac{A}{2} \leq \cot \frac{B}{2} \leq \cot \frac{C}{2}$ and $\sin A \geq \sin B \geq \sin C$

$$\begin{aligned} \sum \frac{\cot^n \frac{B \sin C}{2 \sin A} + \cot^n \frac{C \sin B}{2 \sin A}}{\sin C + \sin B} &\stackrel{\text{Chebyscv}}{\geq} \sum \frac{\frac{1}{\sin A} \left(\frac{1}{2} \left(\cot^n \frac{B}{2} + \cot^n \frac{C}{2} \right) (\sin C + \sin B) \right)}{\sin C + \sin B} = \\ &= \frac{1}{2} \sum \frac{1}{\sin A} \left(\cot^n \frac{B}{2} + \cot^n \frac{C}{2} \right) \stackrel{\text{AM-GM}}{\geq} \frac{1}{2} \sum \frac{1}{\sin A} 2 \sqrt{\left(\cot^n \frac{B}{2} \cot^n \frac{C}{2} \right)} = \\ &= \sum \frac{1}{\sin A} \sqrt{\left(\cot^n \frac{B}{2} \cot^n \frac{C}{2} \right)} \stackrel{\text{AM-GM}}{\geq} 3 \sqrt[3]{\frac{\prod \cot^n \frac{A}{2}}{\prod \sin A}} = 3 \sqrt[3]{\frac{\left(\frac{s}{r}\right)^n}{\frac{sr}{2R^2}}} = 3 \sqrt[3]{2 \left(\frac{R}{r}\right)^2 \left(\frac{s}{r}\right)^{n-1}} \stackrel{\text{Euler}}{\geq} \\ &3 \sqrt[3]{8} \left(\frac{s}{r}\right)^{\frac{n-1}{3}} \stackrel{\text{Mitrinovic}}{\geq} 6(\sqrt{3})^{n-1} = 6 \cdot 3^{\frac{n-1}{2}} \end{aligned}$$

Equality holds for $A = B = C$