ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{m_a+m_b}{w_b+w_c}+\frac{w_b+w_c}{h_c+h_a}+\frac{h_c+h_a}{m_a+m_b}\le 9.\left(\frac{7}{8}.\left(\frac{R}{r}\right)^3-6\right)$$

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$$\frac{m_a+m_b}{w_b+w_c}+\frac{w_b+w_c}{h_c+h_a}+\frac{h_c+h_a}{m_a+m_b}\le\frac{m_a+m_b}{h_b+h_c}+\frac{m_b+m_c}{h_c+h_a}+\frac{m_c+m_a}{h_a+h_b}$$

$$=\sum_{cyc}\frac{m_b+m_c+m_a-m_a}{h_c+h_a}\le(4R+r)\sum_{cyc}\frac{1}{h_c+h_a}-\sum_{cyc}\frac{m_a}{h_c+h_a}$$

$$\le 2R(4R+r)\sum_{cyc}\frac{1}{b(c+a)}-\sum_{cyc}\frac{h_a}{h_c+h_a}$$

$$=2R(4R+r)\sum_{cyc}\frac{ca\left(b^2+\sum_{cyc}ab\right)}{abc(a+b)(b+c)(c+a)}-\sum_{cyc}\frac{bc}{ab+bc}\overset{\text{Cesaro}}{\le}$$

$$\frac{2R(4R+r)}{8.16R^2r^2s^2}\left(abc\sum_{cyc}a+\left(\sum_{cyc}ab\right)^2\right)-\sum_{cyc}\frac{c^2}{ac+c^2}\overset{\text{Bergstrom}}{\le}$$

$$\frac{8R(4R+r)}{8.16R^2r^2s^2}\sum_{cyc}a^2b^2-\frac{4s^2}{s^2+4Rr+r^2+2(s^2-4Rr-r^2)}\overset{\text{Goldstone}}{\le}$$

$$\frac{8R(4R+r).4R^2s^2}{8.16R^2r^2s^2}-\frac{4s^2}{3s^2-4Rr-r^2}=$$

$$\frac{R(4R+r)}{4r^2}-\frac{4s^2}{3s^2-4Rr-r^2}\overset{?}{\le}9.\left(\frac{7}{8}.\left(\frac{R}{r}\right)^3-6\right)$$

$$\Leftrightarrow(63R^3-24R^2r-6Rr^2-400r^3)s^2\underset{(*)}{\overset{?}{\ge}}$$

$$r\left(84R^4-11R^3r-16R^2r^2-578Rr^3-144r^4\right)$$

$\boxed{\text{Case 1}}$ $63R^3-24R^2r-6Rr^2-400r^3\ge 0$ *and then* :

$$(63R^3-24R^2r-6Rr^2-400r^3)s^2\overset{\text{Gerretsen}}{\ge}$$

$$(63R^3-24R^2r-6Rr^2-400r^3)(16Rr-5r^2)$$

$$\overset{?}{\ge}r\left(84R^4-11R^3r-16R^2r^2-578Rr^3-144r^4\right)$$

$$\Leftrightarrow 231t^4-172t^3+10t^2-1448t+536\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(231t^3+290t^2+456t+134(t-2)\right)\overset{?}{\ge}0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \therefore (*) \text{ is true}$$

$\boxed{Case\ 2}$ $63R^3 - 24R^2r - 6Rr^2 - 400r^3 < 0$ and then :

$$\left(63R^3 - 24R^2r - 6Rr^2 - 400r^3\right)s^2$$

$$= -\left(-\left(63R^3 - 24R^2r - 6Rr^2 - 400r^3\right)\right)s^2 \overset{\text{Gerretsen}}{\geq}$$

$$-\left(-\left(63R^3 - 24R^2r - 6Rr^2 - 400r^3\right)\right)\left(4R^2 + 4Rr + 3r^2\right)$$

$$\overset{?}{\geq} r\left(84R^4 - 11R^3r - 16R^2r^2 - 578Rr^3 - 144r^4\right)$$

$$\Leftrightarrow 63t^5 + 18t^4 + 20t^3 - 420t^2 - 260t - 264 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left(63t^4 + 144t^3 + 308t^2 + 196t + 132\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\therefore (*)$ is true $\therefore$ combining cases 1 and 2, $(*)$ is true $\forall \Delta$ ABC

$$\therefore \frac{m_a + m_b}{w_b + w_c} + \frac{w_b + w_c}{h_c + h_a} + \frac{h_c + h_a}{m_a + m_b} \leq 9.\left(\frac{7}{8}.\left(\frac{R}{r}\right)^3 - 6\right)$$

$\forall \Delta$ ABC, "=" iff Δ ABC is equilateral (QED)